Derek Sorensen

Introduction to Homotopy Groups of Spheres

Introduction

In this paper we will define a seqeuence of positively graded groups associated to any topological space X, and called the homotopy groups of X. These come as the natural, higher degree analogues of the familiar fundamental group, and in degrees higher than 1 are abelian. Analogous to the fundamental group of a space X being denoted $\pi_1(X, x_0)$, the notation for the i^{th} homotopy group of X is $\pi_i(X, x_0)$. However, as opposed to other topological invariants, such as homology groups, homotopy groups can be extremely difficult to compute; in exchange, however, they give a great deal of information—in particular about CW-complexes.

For this reason we will focus our attention only on the homotopy groups of spheres, $\pi_i(S^n)$. In contrast to the homology groups, these are surprisingly nontrivial, and even a little bit chaotic. They are also not as easy to compute as the corresponding homology groups. In fact, they are so difficult that they have not all been discovered!

We will begin in Section 1 with the basic definitions of homotopy groups, and show some basic facts about them. In Section 2 we will characterize all homotopy groups of S^1, which is the only sphere for which calculations of the homotopy groups can be done with basic knowledge of covering spaces.

We will then characterize some of the higher homotopy groups of in order of difficulty. In Section 3, we will show that $\pi_i(S^n) \cong 0$ if $i < n$. In Section 5, we will proceed to the case that $i = n$ and show that $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$. And finally, in Section 6 we will discuss the much more copmlicated case of $\pi_i(S^n)$ when $i > n$. This results in two theories: stable and unstable homotopy groups. We will concentrate only on the stable homotopy groups, and in Section 6.2 we will show the amazing result that stable homotopy groups form a graded commutative ring.

Throughout this document, we will assume that the reader is familiar with the material covered in a standard introductory course on algebraic topology. This includes topological (co)homology theory and standard results about the fundamental group and covering spaces. As we move through the material, we will closely follow Chapter 4 of Allen Hatcher's book *Algebraic Topology*. The proofs of most major theorems given in this document follow the general logic of Hatcher's proofs, but are not identical to them.

1 Definitions and Important Concepts

We will use the standard notation of f_t for a homotopy $F : [0,1] \times X \to Y$, where $f_t(x) := F(t, x)$ for all $x \in X$. We will also denote (X, A) to be a topological pair, where X is a topological space and $A \subseteq X$ is a subspace of X. A map of pairs, written $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ such that $f(A) \subseteq B$. Also, if X is a topological space with subspace A, then a homotopy rel A is a homotopy $f_t : X \to Y$ such that for all $a \in A$, $f_t(a)$ is a constant map while varying over t. Finally, unless otherwise specified we will use the conventional CW-structure on S^n, which is one $0-$cell and one $n-$cell, where the gluing map identifies the entire boundary of the $n-$cell to the $0-$cell (here, of course, $n > 0$).

With these definitions and conventions, we are ready to move on to the definition of homotopy groups.

Definition 1.1. *Let (X, x_0) be a topological pair, and let $s_0 \in S^n$ for $n > 0$. Then the* $\boldsymbol{i^{th}}$ ***homotopy group of X*** *with basepoint x_0, written $\pi_i(X, x_0)$, is the set of homotopy classes of maps $(S^n, s_0) \to (X, x_0)$.*

Proposition 1.1. *$\pi_i(X, x_0)$ is a group for $i > 0$, and is abelian for $i \geq 2$*

Proof. Let $[f]$ and $[g]$ be two homotopy classes of maps in $\pi_i(X, x_0)$. We will define a natural operation which generalizes concatonation, the operation of the fundamental group. Recall that for two loops γ and σ in $\pi_1(X, x_0)$, we obtain the loop $\gamma \cdot \sigma$ taking $\gamma(2t)$ on the first half of the unit interval, and then $\sigma(2t - \frac{1}{2})$ on the second half. This gives a new loop centered at x_0 which we call $\gamma \cdot \sigma$, and which is well-defined on equivalence classes (any homotopy h_t between γ and γ' can simply be applied to $\gamma \cdot \sigma$ by h_{2t} on the first half of the unit interval, and the identity map on the second half to give that $\gamma \cdot \sigma \sim \gamma' \cdot \sigma$).

Thus to define $f \cdot g$ in $\pi_i(X, x_0)$, we simply perform a the analogous action of mapping $S^i \xrightarrow{\varphi} S^i \vee S^i \xrightarrow{f \cdot g} X$ where the map φ simply collapses the $(i-1)-$dimensional "equator" of S^i (chosen to contain s_0), and the map $f \cdot g$ applies f to the northern hemisphere, and g to the southern hemisphere in such a way that the wedge point is mapped to x_0. This operation is well-defined on homotopy classes for the same reason as the one-dimensional case given above, is clearly associative and has identity the constant map $S^i \mapsto x_0$. Finally, each map f has inverse $f \circ -id$, where $-id$ is the antipodal map, and thus $\pi_i(X, x_0)$ is indeed a group.

There is a simple, geometric argument which shows that $\pi_i(X, x_0)$ is abelian for $i \geq 2$ by considering a a map $(S^i, s_0) \to (X, x_0)$ equivalently as a map $(I^i, \partial I^i) \to (X, x_0)$. Then when we consider $[f] \cdot [g]$ as we did above, we subdivide I^i in two halves. While maintaining homotopy equivalence, we can continuously shrink these two halves in I^i, mapping everything in their complement to x_0, and then continuously exchange places of both parts. Then if we enlarge these halves to, again, fill all of I^n, we have a homotopy equivalence between $[f] \cdot [g]$ and $[g] \cdot [f]$, and so $\pi_i(X, x_0)$ is abelian (for a slightly more detailed argument, included with pictures, see [1], pp.340-41). □

It is simple to verify that if X is path connected, then $\pi_i(X, x_0)$ is independent of choice of x_0. To show this, let $x_0, x_1 \in X$ and let $\gamma : I \to X$ be a path such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Consider $f : (I^i, \partial I^i) \to (X, x_0)$. While maintaining homotopy equivalence, we can define a map $f : (I^{i'}, \partial I^{i'}) \to (X, x_0)$, where $I^{i'}$ is obtained from I^i by homeomorphically enlarging I^i along straight lines from any point in the interior of I^i, in such a way that each point on ∂I^i is uniquely connected to a point on the boundary of $I^{i'}$ by a straight line homeomorphic to $[0, 1]$. Then we can simply define the image of each of these lines to be the image under the path γ, which will gives us $f_\gamma : (I^{i'}, \partial I^{i'}) \to (X, x_1)$ which sends the boundary of $I^{i'}$ to x_1 instead of x_0. Since by construction $I^{i'} \cong I^i$, these two maps are homotopic, and thus if either f_γ or f is homotopic to another map g, the same is true for f or f_γ, respectively. So the homotopy class of f is in $1-1$ correspondence with the homotopy class of f_γ. Note further that this operation is compatible with the group structure on both $\pi_i(X, x_0)$ and $\pi_i(X, x_1)$. Since this is true for all maps, and so $\pi_i(X, x_0) \cong \pi_i(X, x_1)$ for all $x_0, x_1 \in X$.

Thus in the case that X is path connected, we will simply write $\pi_i(X)$. In particular, we will write $\pi_i(S^n)$ for the i^{th} homotopy group of S^n.

Additionally, note that for $i < 0$, based on our given definitions, $\pi_i(X)$ doesn't make any sense. Also, $\pi_0(S^n)$ gives no interesting information about S^n, as every map $S^0 \to S^n$ is nullhomotopic. Furthermore, for any $n \geq 1$, the set of homotopy classes of maps $S^n \to S^0$ contains two elements which cannot relate to each other in any reasonable way, and which do not make particular sense by the definitions above. If we desire to endow π_0 with a group structure, our definitions would give that $\pi_n(S^0) \cong 0 \times 0 \cong 0$, $\pi_0(S^n) \cong 0$ for $n > 0$, and $\pi_0(S^0) \cong 0 \times 0 \times 0 \times 0 \cong 0$. The interesting groups begin at $i = 1$, and thus our treatment of homotopy groups will only focus on $\pi_i(S^n)$ where $i, n \geq 1$.

Our final observation before moving on to relative homotopy groups is that $\pi_i(X)$ is functorial in X, in that a continuous map $\varphi : X \to Y$ induces a map $\varphi_* : \pi_i(X) \to \pi_i(Y)$ for all i via composition with ϕ, i.e. $(S^i \to X) \mapsto (S^i \to X \xrightarrow{\varphi} Y)$. This is, of course, well-defined on homotopy classes (if $f \sim f'$, then $(f \circ g) \sim (f' \circ g)$ and $(g \circ f) \sim (g \circ f')$ via essentially the same homotopy) and also a group homomorphism, as $\varphi \circ (f + g) \sim (\varphi \circ f) + (\varphi \circ g)$. The identity map is sent to the identity, as well as compositions to compositions. Thus $\pi_i(-)$ is a functor from topological spaces to (abelian if $i \geq 2$) groups.

1.1 Relative Homotopy Groups

We will now introduce a very useful notion that will be used for some of the theorems we will prove in Section 6.1. This is the notion of **relative homotopy groups.** There are two equivalent ways of defining them, which we will present below.

For the first, consider I^n, where $I = [0, 1]$, and I^{n-1} be the $(n-1)-$face of I^n which is zero in the last coordinate. Then define J^{n-1} to be the union of all the $(n-1)-$faces of I^n except I^{n-1}. Equivalently, J^{n-1} is the closure of $\partial I^n \setminus I^{n-1}$. Then we have a definition:

Definition 1.2. *Let (X, A) be a topological pair and let $x_0 \in A$. The **relative homotopy group** of the pair (X, A), written $\pi_n(X, A, x_0)$, is the set of homotopy classes of maps of the form $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$.*

Note that this definition allows us to use the same sum operations that we had before, and if A is connected, is independent of basepoint x_0, and so in that case we can simply write $\pi_n(X, A)$ (in our case we will concern ourselves only with maps of spheres and connected subspaces of spheres, and so we will omit the basepoint in all of our calculations).

The most obvious alternate and equivalent definition of homotopy groups of spheres is as follows

Definition 1.3. *Let (X, A) be a topological pair. Then the n^{th} relative homotopy group of (X, A), denoted $\pi_n(X, A)$, is the set of homotopy classes of maps of pairs $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ with the group operation on $\pi_n(X, A)$ defined analagously as from Definition 1.2.*

Note that by these definitions, a map $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ or $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents the zero class in the relative homotopy group if it is homotopic rel A to a map whose image lies entirely in A.

Of course, we regain the absolute homotopy definition by letting $A = x_0$ and noting that the definitions are equivalent. Thus, absolute homotopy groups are just a special case of relative homotopy groups.

One of the most helpful and enlightening contributions of relative homotopy groups is that they give us long exact sequences, which are extremely useful for calculating homotopy groups. The long exact sequence we present here will be used in the proof of Theorem 6.1.

Lemma 1.1. *Let (X, A) be a topological pair, and let $x_0 \in A$. Then the following sequence is exact:*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0)$$

where i_* and j_* are the maps induced by the inclusions $i : (A, x_0) \hookrightarrow (X, x_0)$ and $j : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$ and the boundary map ∂ is given by restricting a map $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ to S^{n-1} (or, equivalently, $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to I^{n-1}). This gives a map $\pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ for all $n \geq 2$.

And with just a little more work, the proof for Lemma 1.1 yields the following:

Lemma 1.2. *Let $(X, A, B,)$ be a topological triple, meaning that $B \subseteq A \subseteq X$. Also, let $x_0 \in B$. Then the following sequence is exact:*

$$\cdots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \to \cdots \to \pi_1(X, A, x_0)$$

with i_* and j_* the obvious analogues of Lemma 1.1

In favor of proving more central statements to the theory of homotopy groups of spheres, we will not give these proofs in this document. However, we refer the reader to Theorem 4.3 of [1] for a complete proof.

2 The Homotopy Groups of S^1

We will begin our analysis of homotopy groups of spheres by characterizing the homotopy groups of S^1. As opposed to higher-dimensional spheres, all the homotopy groups $\pi_i(S^1)$ can be computed using elementary theory of the fundamental group from an introductory algebraic topology course. It becomes much more difficult to compute these groups for higher-dimensional spheres.

Proposition 2.1.

$$\pi_i(S^1) \cong \begin{cases} \mathbb{Z}, & i = 1 \\ 0, & i > 1 \end{cases}$$

Proof. It is a basic result from elementary algebraic topology that $\pi_1(S^1) \cong \mathbb{Z}$. So we only need to show that for all $i > 1$, $\pi_i(S^1) \cong 0$.

This follows immediately because for all $n > 1$, S^n is simply connected, and so $\pi_1(S^n) \cong 0$. Thus, any map $f : S^n \to S^1$ induces a map $f_* : \pi_1(S^n) \to \pi_1(S^1)$ which is the unique map $f_* : 0 \to \mathbb{Z}$. Since the universal cover of S^1 is $\mathbb{R}$, and $\pi_1(\mathbb{R}) \cong 0$, the image of f_* lies in the image of p_*, where $p : \mathbb{R} \to S^1$ is the projection of the universal cover. Thus f can be lifted to a map $g : S^n \to \mathbb{R}$ such that $f = p \circ g$. But $\mathbb{R}$ is contractible, and thus g is nullhomotopic, giving that $f = p \circ g$ is also nullhomotopic. $\square$

You'll note that the key element of this proof was the fact that S^1 has a contractible universal cover. But this is false for spheres of finite dimension higher than 1 because for $n \geq 2$, S^n is simply connected, and thus S^n is its own universal cover. We need a little more machinery in order to calculate $\pi_i(S^n)$, $n \geq 2$, as even at $n = 2$ we get highly nontrivial homotopy groups.

3 Homotopy Groups $\pi_i(S^n)$ for $i < n$

We will begin looking at higher dimensional spheres with the case that $i < n$. It turns out that all groups of this form are trivial. This might match the reader's intuition because for all n, S^{n+1} is a suspension of S^n, and thus the subspace of S^{n+1} corresponding to S^n can be continuously deformed to a point. So, at a glance it would seem to be the case that any map $S^n \to S^{n+1}$ should be homotopic to a constant map.

This might sound trivial, but, while this result holds, it does not come for free. We might even hope that we could prove something to the effect that any map $S^n \to S^k$, with $n < k$, is not surjective. If that held, then we could choose a point in S^k which is not in the image, and note that its complement is homeomorphic to D^k, which is contractible, and we would have our result. While any map $S^n \to S^k$ is homotopic to a nonsurjective map, a surjection $S^1 \twoheadrightarrow S^k$, and similarly a surjection $S^n \twoheadrightarrow S^k$, can be constructed with space-filling curves.

We will need a much stronger result, which is quite remarkable. Before introducting the result, we will need the following notion:

Definition 3.1. *Let X, Y be CW-complexes, where X^n and Y^n denote the $n-$skeletons of X and Y, respectively. Then a map $f : X \to Y$ is* ***cellular*** *if, for all n, $f(X^n) \subseteq Y^n$.*

Of course, for all i, S^i is a CW-complex with one $0-$cell and one $i-$cell. So if we could show that, for $n < k$, any map $S^n \to S^k$, is homotopic to a cellular map, then we would immediately have our result because the $n-$skeleton of S^k is a single point.

As it turns out, this result holds for *any* continuous map between CW-complexes. This theorem is called the Cellular Approximation Theorem, and is quite general in scope. Regardless, will need the theorem in its most general form in order to prove some of our results further on. So, we will prove it and get that if $i < n$, $\pi_i(S^n) \cong 0$ as a corollary.

The proof becomes very straightforward with the help of the following technical lemma.

Lemma 3.1. *Let $f : I^n \to Z$ be a map, where Z is obtained from a subspace W by attaching a cell e^k. Then f is homotopic rel $f^{-1}(s)$ to a map f_1 such that there is a $k-$simplex $\Delta^k \subset e^k$ where $f_1^{-1}(\Delta^k)$ is a (possibly empty) union of finitely many convex polyhedra and f_1 restricted to each of them is a linear surjection $\mathbb{R}^n \to \mathbb{R}^k$.*

Note that in the case that $n < k$, there do not exist any linear surjections $\mathbb{R}^n \to \mathbb{R}^k$, and so $f_1^{-1}(\Delta^k) = \emptyset$.

Proof. Consider the isomorphism $e^k \cong \mathbb{R}^k$, and let $B_i \subseteq \mathbb{R}^k$ be a closed ball of radius i centered at the origin. Then since f is continuous, $f^{-1}(B_2)$ is closed, and thus compact in I^n as I^n itself is compact. It is a standard result in introductory analysis courses that a continuous map is uniformly continuous on a compact subset. So f is uniformly continuous on $f^{-1}(B_2)$. Thus there exists some $\epsilon > 0$ such that if $|x - y| < \epsilon$, then $|f(x) - f(y)| < \frac{1}{2}$ for all $x, y \in f^{-1}(B_2)$. Subdivide I such that the induced subdivision I^n has $n-$cubes of diameter less than ϵ. Then let K_1 be the union of all such (closed) cubes which intersect $f^{-1}(B_2)$, and let K_2 be the union of all such cubes which intersect K_1. Then we have obvious inclusions $f^{-1}(B_1) \subset K_1 \subset K_2 \subset f^{-1}(B_2)$, where $K_2 \subset f^{-1}(B_2)$ because each point in $f(K_2)$ has distance less than $\frac{1}{2}$ from any point in $f(K_1)$, which itself has distance less than $\frac{1}{2}$ from B_1. Since the minimum distance from any point of ∂B_2 to any point of ∂B_2 is 1, the inclusion holds.

Now note that we can view K_2 as a CW-complex, where for all $i \leq k$, the $i-$cells of K_2 are the open $i-$faces of the $k-$dimensional cubes of K_2. We can build a simplicial structure on K_2 inductively via the barycentric subdivision of the cubical cells. That is, we can build a simplicial structure by taking the vertices of the simplicial structure to be the center points of the cells and then taking cones of the simplicial structure on the boundary of each cubical cell to the center point of the cell. A simplicial structure of the boundary induces a simplicial structure on the cone, so we have a well-defined simplicial structure on K_2.

Now let $g : K_2 \to e^k \cong \mathbb{R}^k$ be a map that equals f on all vertices of our given simplices and is linear when restricted to each simplex. Then define a map $\varphi : K_2 \to I$ where $\varphi(\partial K_2) = 0$ and $\varphi(K_1) = 1$ (we can clearly build this map because of the distance ∂K_2 from K_1). Then we define a homotopy $f_t : K_2 \to e^k$ via the formula

$$f_t = (1 - t\varphi)f + (t\varphi)g.$$

Of course, $f_0 = f$, and the restrictions of both f_1 and g to K_1 are equal. We can extend f_t by noting that f_t is constant on ∂K_2 and so we can simply extend f_t to the rest of I^n by letting it be constant on the complement of K_2 as well.

As one of our final steps, we will need to show that there is an open neighborhood of 0, $N \subset B_1$, such that $f_1^{-1}(N) \subset K_1$. This is equivalent to showing that $f_1(K_1^c) \subseteq N^c$, where X^c is the complement of X. Since $f_1 = f$ on the complement of K_2, $f_1(K_2^c) \subseteq B_1^c$ and so $f_1(K_2^c)$ is contained in the complement of any such open neighborhood N.

Then for $K_2 \setminus K_1$, consider a simplex σ from the simplicial structure we defined on K_2. This is mapped under f to a ball B_σ with radius $\frac{1}{2}$. Then since B_σ is convex, g also maps σ into B_σ, and therefore so does f_1. In the case that σ is not contained in K_1, B_σ intersects the exterior of B_1 and thus does not contain 0, and so it does not contain a neighborhood N_σ of 0. There are only finitely many such σ's, so we can choose a neighborhood of 0 in B_1, for example $N = \bigcap_{\sigma \not\subseteq K_1} N_\sigma$, which is disjoint from $f_1(\sigma)$ for all $\sigma \not\subseteq K_1$. Then $f_1(K_1^c) \subseteq N^c$, and thus $f_1^{-1}(N) \subseteq K_1$, as desired.

Finally, consider a simplex $\Delta^k \subset N$. Then $f_1^{-1}(\Delta^k) \subset K_1$ is given by the union of its intersections with simplices σ of K_1. Defining $L_\sigma : \mathbb{R}^n \to \mathbb{R}^k$ to be the linear map given by restricting g onto σ, each of these intersections is the intersection of σ with $L_\sigma^{-1}(\Delta^k)$. Since $L_\sigma^{-1}(\Delta^k)$ is a convex polyhedron, each of these intersections is a convex polyhedron. So to choose our desired Δ^k, we need only consider the nonsurjective L_σ's and choose Δ^k to be in the complement of the image of each L_σ. This is, of course, possible because the image of each L_σ is just the union of finitely many hyperplanes of dimension less than k.

Thus we have a Δ^k which satisfies the conditions of the lemma, and we have our result. $\square$

Now with this lemma, we can inductively homotope any map $f : X \to Y$ of CW-complexes to a cellular map, as follows:

Theorem 3.2. *(Cellular Approximation Theorem) Let X and Y be CW-complexes. Then any map $f : X \to Y$ is homotopic to a cellular map.*

Proof. First, note that any map $f : X \to Y$ is homotopic to a map which is cellular on the $0-$skeleton. This is because every path component of Y has a $0-$cell, and any point in Y is path-connected to a $0-$cell (we can construct a path by simply moving from a point in Y to the boundary of a cell which contains it; once at the boundary, we can continue to the boundary of that cell, etc. This process must terminate at a zero cell, so we can build our path). Thus we have paths from the image of each $0-$cell of X to the $0-$cells of Y, which then by the homotopy extension property gives us a homotopy of f defined on all of X which makes f cellular. So we may assume that f is cellular on X^0.

Then for our inductive step, suppose that $f : X \to Y$ is cellular on X^{n-1}, and let e^n be an $n-$cell in X. Since the closure of I^n is compact, the closure of e^n is compact in X, and thus its image under f is also compact. Thus $f(e^n)$ only intersects finitely many cells in Y, so we can consider e^k, a cell in Y that intersects $f(e^n)$ such that any other cell intersecting $f(e^n)$ has dimension $\leq k$. If $k \leq n$, then f is already cellular on e^n, and so we are done with this cell. For the other case, consider $k > n$.

Now consider the map $f : X^{n-1} \cup e^n \to Y^k$, and, in particular, its composition with the characteristic map $I^n \to X$ for e^n. Then the resulting map is as in the lemma, where $Z = Y^k$ and $W = Y^k \setminus e^k$. The lemma gives us a homotopy which fixes ∂I^n, and thus induces a homotopy f_t of $f|X^{n-1} \cup e^n$ fixed on X^{n-1}. Since $k > n$, the preimage of the $k-$simplex Δ^k given by the lemma is empty. Then f is homotopic to a map which does not surject on e^k, and thus we can homotope $f|X^{n-1} \cup e^n$ so that its image is disjoint from e^k.

After finitely many iterations of using the result of Lemma 3.1, we can produce a homotopy which ensures that all the cells which $f(e^n)$ intersects of dimension strictly higher than n are missed, and thus f maps e^n into the $n-$skeleton of Y.

We can repeat this procedure for each $n-$cell of X e^n_α, where α is in some (possibly infinite) indexing set I, and get a homotopy of $f|X^n$ which results in a cellular map. Note that, by construction, this homotopy is stationary on both X^{n-1} and A^n, and so we have a homotopy rel $X^{n-1} \cup A^n$ producing a cellular map. Then by the homotopy extension property, this homotopy of $f|X^{n-1}$ can be extended to a homotopy of f, and so we have a resulting homotopy $f_t^{(n)}$ of f which produces a map $f_1^{(n)}$ which is cellular on the $n-$skeleton of X.

We continue this process and obtain a (possibly infinite) sequence of homotopies

$$f_t^{(n)}, f_t^{(n+1)}, f_t^{(n+2)}, \ldots$$

where $f_t^{(k)}$ is the homotopy of $f_t^{(k-1)}$ which produces a map which is cellular on X^k. From these we can construct a homotopy f_t. If X is finite-dimensional, with dimension k, then we can simply subdivide $[0, 1]$ into k parts and homotope f using $f^{n+l}_{(k-n)t}$ on the l^{th} subdivision of $[0, 1]$. If X is not finite-dimensional, then we can do the same, but by dividing $[0, 1]$ into intervals $[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$.

Thus f is homotopic to a cellular map, as desired, and we have our result. □

Corollary 3.2.1. *$\pi_n(S^k) \cong 0$ for all $n < k$.*

Proof. If $n < k$, any map $S^n \to S^k$ is nullhomotopic by Theorem 3.2 and the standard CW-structure on S^i, and thus 0 in $\pi_n(S^k)$. □

4 CW Approximation

In the spirit of Theorem 3.2, we will also include a brief introduction to CW-approximation. The main result of this section does not have any immediate applications to homotopy groups of spheres, but turns out to be important later on, in particular for the proof of Theorem 6.1. The most central idea of CW approximation is that CW-complexes are so convenient, for several reasons (including Theorem 3.2). So if possible, we would like to work with CW-complexes, even while working on homotopy groups of general spaces. Thus for any space X, we want to find a CW-complex that resembles X well-enough to gather with the homotopy groups of X. This is done in the form of a CW-approximation, which is a space that not only has isomorphic homotopy groups to X, but also comes with a map into X that induces the isomorphisms (these are two distinct conditions!). The main, remarkable result is that for any n we can find such an approximation for *any* topoloical space X.

We introduce the main result with the following definitions:

Definition 4.1. *Let X be a topological space. Then X is* ***n−connected*** *if $\pi_i(X) \cong 0$ for all $i \leq n$. Similarly, a pair (X, A) is $n-$connected if $\pi_i(X, A) \cong 0$ for all $i \leq n$.*

The condition that $\pi_i(X, A) \cong 0$ for all $i \leq n$ is equivalent to saying that for $i \leq n$, any map $S^i \to X$ is homotopic rel A to a map whose image is entirely contained in A. Also, by Corollary 3.2.1, S^n is $(n-1)-$connected for all $n > 1$.

Definition 4.2. *Let (X, A) be a topological pair. Then an* ***n−connected CW model for (X, A)*** *is an $n-$connected CW pair (Z, A) with a map $(Z, A) \to (X, A)$ which restricts to the identity on A and such that the induced map $\pi_i(Z) \to \pi_i(X)$ is an isomorphism for $i > n$ and an injection for $i = n$.*

We are finally prepared to give the main result of this section:

Theorem 4.1. *Let (X, A) be a topological pair, where A is a nonempty CW-complex. Then there exist $n-$connected CW models $f : (Z, A) \to (X, A)$ for all $n \geq 0$, such that Z is obtained from A by attaching cells of dimension greater than n.*

Proof. We will give a construction for Z as a union of subcomplexes begining with A and then increasing in a chain $A = Z_n \subseteq Z_{n+1} \subseteq \cdots$ where Z_k is obtained from Z_{k+1} by attaching $k-$cells. By induction, suppose that we have Z_k for some k and a map $f : Z_k \to X$ which restricts to the identity on A, and where f_*, the induced map on π_i, injects for $n \leq i \leq k$ and surjects for $n < i \leq k$ with respect to a choice of basepoint $x_\gamma \in A_\gamma$, the components of A. Our base case, $k = n$, vacuously satisfies these conditions.

Now for the inductive step, we choose generators of the kernal of $f_* : \pi_k(Z_k, x_\gamma) \to \pi_k(X, x_\gamma)$ for all γ, which are cellular maps $\varphi_\alpha : S^k \to Z_k$. Let Y_{k+1} be the complex obtained by attaching cells e_α^{k+1} to Z_k via our maps φ_α. By functoriality of π_i, the compositions $f \circ \varphi$ are nullhomotopic, so we can extend f over Y_{k+1}. Thus the resulting map $f_* : \pi_k(Y_{k+1}, x_\gamma) \to \pi_k(X, x_\gamma)$ injects, because any element of the kernel of f^* is a cellular map, and is nullhomotopic by construction ($f \circ \varphi$ is nullhomotopic). Also, because $\pi_k(Z_k) \to \pi_k(Y_{k+1})$ surjects, the composition $\pi_k(Z_k) \to \pi_k(Y_{k+1}) \to \pi_k(X)$ also surjects. Furthermore, the homotopy groups π_i for $i < k$ are not affected by attaching cells e_α^{k+1} because $i < k < k + 1$. Because π_0 has no reasonable group structure, in the case that $k = 0$ we instead construct Y_1 by attaching $1-$cells which join all basepoint $0-$cells x_γ which lie in the same path component of X.

Now choose maps $\psi_\beta : S^{k+1} \to X$ which generate $\pi_{k+1}(X, x_\gamma)$ for all γ. Define Z_{k+1} to be the wedge of Y_{k+1} with spheres (indexed by β) S_β^{k+1}, which are wedged at the basepoints x_γ corresponding to each β. Then extend f to Z_{k+1} by letting f equal ψ_β on each S_β^{k+1}. This gives us that the map $f_* : \pi_{k+1}(Z_{k+1}, x_\gamma) \to \pi_{k+1}(X, x_\gamma)$ surjects. Thus the induced map of the inclusion map $Y_{k+1} \hookrightarrow Z_{k+1}$ surjects by cellular approximation, and surjects because Z_{k+1} retracts onto Y_{k+1}. Thus the induced map is an isomorphism, which finishes the inductive step.

Finally, note that the maps $f_* : \pi(Z, x_\gamma) \to \pi_i(X, x_\gamma)$ depend only on the $(i+1)-$skeleton of Z by construction, and so they are isomorphisms for all $i > n$ and injective on $i = n$. This is independent of choice of x_γ, as any point in Z can be joined a path to some x_γ. □

This gives us a corollary which turns out to be key to some of our proofs later on.

Corollary 4.1.1. *Let (X, A) be an $n-$connected CW-pair. Then there exists a CW pair (Z, A) such that $Z \simeq X$ rel A where $Z \setminus A$ has cells only in dimension greater than n.*

Proof. Consider the pair (Z, A) and map $f : (Z, A) \to (X, A)$ from Theorem 4.1. We first need to check that $Z \simeq X$. Of course, by choice and construction, f induces isomorphisms $\pi_i(Z) \to \pi_i(X)$ for all $i > n$ and an injection on $i = n$. Also, by construction the inclusions $A \hookrightarrow Z$ and $A \hookrightarrow X$ induce isomorphisms in degrees $i < n$ and a surjection $i = n$. So f induces an isomorphism for all n. Our homotopy equivalence comes from Theorem 4.5 of [1], p.346.

As X and Z are CW-complexes, f is thus a homotopy equivalence. We now only need to check that $X \simeq Z$ rel A. To do so, we will use M_f, the mapping cylinder of f. Let W be the quotient of M_f which collapses each segment $\{a\} \times I$, where $a \in A$. Then homotope f to be cellular (by cellular approximation). Thus W is a CW-complex containing both X and Z as subcomplexes. As M_f deformation retracts onto X, so does W. Furthermore, all groups $\pi_i(W, Z) = 0$ because f induces isomorphisms on all homotopy groups $\pi_i(W) \cong \pi_i(Z)$. Thus W also deformation retracts onto Z. Both of these deformation retracts from W onto X and Z are the identity on A, and thus we have that $X \simeq Z$ rel A. □

5 Homotopy Groups $\pi_i(S^n)$ for $i = n$

We will now progress in complexity of the homotopy groups $\pi_i(S^n)$ to the case where $i = n$. As in the case that $i < n$, the result that $\pi_n(S^n) \cong \mathbb{Z}$ is intuitively obvious. Indeed, with the tools of an introductory algebraic topology course, this result comes quickly, as follows:

Proposition 5.1. *$\pi_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$.*

Proof. Note that any nonsurjective map can be homotoped to a constant map, because the complement of a point is contractible in the sphere. So to calculate the nontrivial classes of maps $f : S^n \to S^n$ in $\pi_n(S^n)$, we only need to look at the case that f surjects. But each surjective map has a well-defined degree, $k \in \mathbb{Z} \setminus \{0\}$, which is unique up to homotopy. By the definition given for concatonation of maps, if $[f]$ is of degree l and $[g]$ is of degree k, then $[f] \cdot [g]$ is of degree $l + k$. So the map $\pi_n(S^n) \to \mathbb{Z}$ which maps every map to its degree is well-defined. It injects by the arguments above, and surjects because $\pi_n(S^n)$ is generated by the identity map, which as degree 1, which has inverse the antipodal map, which has degree -1. So we have that $\pi_n(S^n) \cong \mathbb{Z}$, as desired. □

This can be proved in a few different ways—we will even give an alternate, nearly identical proof of this theorem in Section 6, as Corollary 6.1.1.

However, there is a result with which we can prove that $\pi_n(S^n) \cong Z$ which only makes use of the *homology groups* of S^n and the fact that S^n is $(n-1)-$connected. This remarkable result, called the Hurewicz theorem, hints at a much deeper and beautiful connection to (co)homology than we can express in this short paper. Although we will not be able to show here, there is a reformulation of the cohomology of a space X which uses homotopy classes of maps $X \to K(G,n)$, where $K(G,n)$ is a space which admits only one nontrivial homotopy group, G, in degree n (see Section 4.3 of [1]).

For this reason, we will simply state the Hurewicz theorem without proof, and refer the curious reader to Theorem 4.32 of [1] for a proof.

Theorem 5.1. *(Hurewicz theorem) Let X be $(n-1)-$connected where $n \geq 2$. Then $\tilde{H}_i(Z) \cong 0$ for $i < n$, and $\pi_n(X) \cong H_n(X)$.*

As S^n is $(n-1)-$connected by Corollary 3.2.1, we immediately have:

Corollary 5.1.1. $\pi_n(S^n) \cong \mathbb{Z}$.

6 Homotopy Groups $\pi_i(S^n)$ for $i > n$

We will now move on to the much more complicated case of $\pi_i(S^n)$, where $i > n$. It might come as a surprise to many readers that these groups are highly nontrivial and difficult to compute, especially since $H_i(S^n) \cong 0$ if $i > n$—and in general, the i^{th} homology groups of any $n-$dimensional manifold are always 0 if $i > n$.

Even intuitively it seems plausible that a map $S^i \to S^n$ might be nullhomotopic in general if $i > n$. But as we find out with the higher homotopy groups of spheres, this is definitely not the case; these maps behave and vary in a much more chaotic fashion than one would intuitively expect [2].

Example 6.1. *(Hopf fibration) One of the more interesting, motivating examples for this section is the Hopf fibration (or the Hopf bundle), which is a map $f : S^3 \twoheadrightarrow S^2$ not homotopic to the identity. It's a good example of a map of nontrivial homotopy type from a higher dimensional sphere onto S^2.*

We can most easily see f by considering it as the projection map of the fiber bundle $S^1 \hookrightarrow S^3 \xrightarrow{f} S^2$. Here we see S^1 as the group of unit-length complex numbers, and S^3 as the group of unit-length quaternions. Then $f : S^3 \twoheadrightarrow S^2$ is best understood as the cokernel of the inclusion $S^1 \hookrightarrow S^3$ of $S^1 \leq \mathbb{C}$ into $S^3 \leq \mathbb{H}$.

These groups are extremely difficult to compute, and so we will not be able to give a sweeping classification like we did in the case that $i \leq n$. We will instead focus on a certain subset of these groups, namely the stable homotopy groups of spheres.

6.1 Stable Homotopy Groups of Spheres

The theory of stable homotopy groups of spheres is as unexpected and remarkable as anything else so far described. It shows that, because an $n-$sphere can be regarded as the suspension of an $(n-1)-$sphere, there is a sequence of isomorphisms

$$\pi_i(S^n) \to \pi_{i+1}(S^{n+1}) \to \pi_{i+2}(S^{n+2}) \to \cdots$$

which begins, depending on i, at a sufficiently large n. The isomorphism classes resulting from this form a *graded ring*, as we will show later on. This is yet another example of the amazing structure of homotopy groups.

The theorem mentioned, called the Freudenthal suspension theorem, is below, and gives us the basis for the theory of stable homotopy groups.

Theorem 6.1. *(Freudenthal suspension theorem) Let X be an $n-$connected topological space. Then the map of homotopy groups induced by the suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n - 1$.*

The result from this theorem is that, for any i and n, the maps

$$\cdots \to \pi_{i-1}(S^{n-1}) \to \pi_i(S^n) \to \pi_{i+1}(S^{n+1}) \to \cdots$$

eventually stabilize to isomorphisms after enough terms, depending on the difference $i - n$. So for any i, we can consider the sequence induced by the suspension maps:

$$\cdots \to \pi_{i+(n-1)}(S^{n-1}) \to \pi_{i+n}(S^n) \to \pi_{i+(n+1)}(S^{n+1}) \to \cdots \tag{1}$$

and note that (1) stabilizes at $n = i + 2$. From then on, any group of the form $\pi_{i+m}(S^m)$, where $m > i + 2$, is isomorphic to $\pi_{2i+2}(S^{i+2})$. We call the group $\pi_{2i+2}(S^{i+2})$ the *stable $i-$stem* and write it π_i^s.

Corollary 6.1.1. $\pi_0^s \cong \mathbb{Z}$

Proof. To prove this result, we only need that $\pi_n(S^n) \cong \mathbb{Z}$ for $n \geq 2$. This was proved in Proposition 5.1, but as promised we will give a slightly more rigorous proof, using Theorem 6.1. Note that Theorem 6.1 gives us a sequence of maps

$$\pi_1(S^1) \xrightarrow{\varphi} \pi_2(S^2) \to \pi_3(S^3) \to \cdots$$

where φ is a surjection, and the rest are isomorphisms. So we know that for $n > 1$, $\pi_n(S^n) \cong \mathbb{Z}/ker(\varphi)$. We also know that $\pi_n(S^n)$ is infinite, because for any n we can construct a map $S^n \to S^n$ of degree k for any $k \in \mathbb{N}$. Thus φ has a trivial kernel, and $\pi_n(S^n) \cong \mathbb{Z}$, as desired. □

Before moving on to the proof of Theorem 6.1 we will need to prove the following proposition, called Excision for Homotopy Groups:

Proposition 6.1. *(Excision for Homotopy Groups) Let X be a CW-complex and let $A, B \subseteq X$ be subcomplexes of X such that $A \cup B = X$ and $C = A \cap B \neq \emptyset$. If (A, C) is $m-$connected and (B, C) is $n-$connected such that $m, n \geq 0$, then the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by the inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.*

Proof. We proceed by cases.
Case 1: First note that A is obtained from C by attaching cells e_α^{m+1}, and B is obtained from C by attaching a single cell e^{n+1}. We will show that $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism, and we will begin by showing surjectivity. To do so, consider a map $f : (I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$. Because I^i is compact, the image of f is compact and therefore intersects only finitely many of the cells e_α^{m+1} and e^{n+1}. Thus by finitely many iterations of applying Lemma 3.1, we can homotope f, through maps of the form $(I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$ (i.e., all the maps still map ∂I^i into B and J^{i-1} to x_0) such that, for simplices Δ_α^{m+1} and Δ^{n+1} corresponding to the cells that the image of f intersects, $f^{-1}(\Delta_\alpha^{m+1})$ and $f^{-1}(\Delta^{n+1})$ are finite unions of convex polyhedra, such that on each f is is the restriction of a linear surjection from $\mathbb{R}^i$ to $\mathbb{R}^{m+1}$ or $\mathbb{R}^{n+1}$.

Then we will need the following fact:

Fact: if $i \leq m + n$, then there exist points $p_\alpha \in \Delta_\alpha^{m+1}$ and $q \in \Delta^{n+1}$, and a map $\varphi : I^{i-1} \to [0, 1)$ such that:

1. $f^{-1}(q)$ lies below the graph of φ in $I^{i-1} \times I = I^i$.
2. $f^{-1}(p)$ lies above the graph of φ for each α.
3. $\varphi = 0$ on ∂I^{i-1}.

For a complete proof of the fact, refer to [1], p.362.

Given this fact, let f_t be a homotopy of f which excises the region under the graph of φ by restricting f to the region above the graph of $t\varphi$ for $t \in [0,1]$. Then let $P = \bigcup_{\alpha} \{p_\alpha\}$ and $Q = \{q\}$, and note that by (2), $f_t(I^{i-1}) \cap P = \emptyset$ and $f_1(I^i) \cap Q = \emptyset$ by (1). Then consider the following diagram:

$$\begin{array}{ccc} \pi_i(A, C) & \longrightarrow & \pi_i(X, B) \\ \cong \big\downarrow & & \big\downarrow \cong \\ \pi_i(X \setminus Q, X \setminus Q \setminus P) & \longrightarrow & \pi_i(X, X \setminus P) \end{array}$$

and note that the class of our map $[f]$ in $\pi_i(X, B)$, when regarded as its isomorphic image in $\pi_i(X, X \setminus P)$, is equal to the element $[f_1]$ from our homotopy f_t, which is in the image of the map $\pi_i(X \setminus Q, X \setminus Q \setminus P) \to \pi_i(X, X \setminus P)$. Thus we have surjectivity.

We now consider injectivity. Let $f_0, f_1 : (I^i, \partial I^i, J^{i-1}) \to (A, C, x_0)$ be representatives of two classes in $\pi_i(A, C)$ such that, under the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by the inclusion $(A, C) \to (X, B)$, they are mapped to the same class in $\pi_i(X, B)$. That gives us a homotopy from f_0 to f_1 of the form of a map $F : [0,1] \times (I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$. Then by deformation, again using Lemma 3.1, we can construct a map $\varphi : I^{i-1} \times I \to [0, 1)$ which separates $F^{-1}(p)$ from the sets $F^{-1}(p_\alpha)$ as in the surjectivity case. Thus, again as before, we can excise $F^{-1}(p)$ from the domain of F, giving us that f_0 and f_1 are two representatives of the same class in $\pi_i(A, C, x_0)$. However, injectivity only holds for $i < m + n$, as opposed to surjectivity, because in the proof of injectivity, $I^i \times I$ plays the role that I^i played in the surjectivity proof, and so now we have the requirement that $i + 1 \leq m + n$, or equivalently that $i < m + n$.

Case 2: We will now prove the proposition for the case that A is obtained from C by attaching cells of dimension $\geq n+1$. To show first the surjectivity $\pi_i(A, C) \to \pi_i(X, B)$, we consider a map $f : (I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$ which is a representative of an equivalence class in $\pi_i(X, B)$. As in Case 1, the image of f is compact, and thus intersects only finitely many cells (by the weak topology), and thus by iterations of Case 1 we can homotope f off the cells of $B \setminus C$ one at a time, beginning from the cell of largest dimension and each time decreasing dimension. For injectivity, the proof is very similar as in Case 1, where we begin with a homotopy $F : [0,1] \times (I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$ and pushing F off cells in $B - C$. Again, for the same reasons, we get an isomorphism for $i < m + n$, but only a surjection for $i = m + n$.

Case 3: In our final case before moving to the general case, we consider the case that A is obtained from C by attaching cells of dimension $\geq m+1$ and B as before, in Case 2. Recall that by cellular approximation (Theorem 3.2) the cells of dimension higher than $m+n+1$ have no effect on π_i if $i \leq m+n$. Thus we can assume without loss of generality that $A \setminus C$ only has cells of dimension $\leq m + n + 1$. Then let $A_k \subseteq A$ be $C \cup A^k$, where A^k is the $k-$skeleton of A, and let $X_k = A_k \cup B$. We will proceed by induction on k, proving the result for maps of the form $\pi_i(A_k, C) \to \pi_i(X_k, B)$.

Our base case is $k = m + 1$, which is Case 2, and so has been proved already. Then for the inductive step, we will consider the commutative diagram with rows formed by the exact sequence of triples (from Lemma 1.2) of (A_k, A_{k-1}, C) and (X_k, X_{k-1}, B):

$$\begin{array}{ccccccccc} \pi_{i+1}(A_k, A_{k-1}) & \longrightarrow & \pi_i(A_{k-1}, C) & \longrightarrow & \pi_i(A_k, C) & \longrightarrow & \pi_i(A_k, A_{k-1}) & \longrightarrow & \pi_{i-1}(A_{k-1}, C) \\ {\scriptstyle 1}\big\downarrow & & {\scriptstyle 2}\big\downarrow & & {\scriptstyle 3}\big\downarrow & & {\scriptstyle 4}\big\downarrow & & {\scriptstyle 5}\big\downarrow \\ \pi_{i+1}(X_k, X_{k-1}) & \longrightarrow & \pi_i(X_{k-1}, B) & \longrightarrow & \pi_i(X_k, B) & \longrightarrow & \pi_i(X_k, X_{k-1}) & \longrightarrow & \pi_{i-1}(X_{k-1}, B) \end{array}$$

and we will apply the $5-$lemma in each case of i to show the desired isomorphism. Note that if $i < m + n$, then (1) and (4) are isomorphisms from Case 2, and (2) and (5) are isomorphisms by the inductive hypothesis. Thus by the 5-lemma, (3) is an isomorphism as well. Then if $i = m + n$, (2) and (4) and (5) is injective, which is all that is needed to show that (3) surjects in the proof of the 5-lemma. We may have a more complicated situation when $i = 2$, as some of the groups may be nonabelian, and $\pi_{i-1}(A_{k-1}, C)$ and $\pi_{i-1}(X_{k-1}, B)$ might not even be groups. But with slight changes to the proof of the 5-lemma, the theorem will still hold (see [1], p.363). Finally, if $i = 1$, we have the result that $\pi_1(A, C) \to \pi_1(X, B)$ is either surjective or an isomorphism because if $m \geq 1$, then $\pi_1(A, C) \cong \pi_1(X, B) \cong 0$, and if $m = 0$, then $n \geq 1$ and the result follows by cellular approximation.

Finally, we will treat the general case. By Corollary 4.1.1, because of the assumptions about connectivity, (A, C) and (B, C) are homotopy equivalent to pairs (A', C) and (B', C) as in Case 3. Furthermore, these homotopy equivalences fix C, and so we can fit these homotopy equivalences together to get a homotopy equivalence $A \cup B \simeq A' \cup B'$, and so the general case reduces to Case 3.

□

From Proposition 6.1, Theorem 6.1 follows in a very straightforward way:

Proof of Theorem 6.1. We can write SX as the union of the two cones of the suspension C_+X and C_-X, where $C_+X \cup C_-X \cong X$. Then the following square commutes:

$$\begin{array}{ccc} \pi_i(X) & \longrightarrow & \pi_{i+1}(SX) \\ {\scriptstyle \cong}\big\downarrow & & {\scriptstyle \cong}\big\downarrow \\ \pi_{i+1}(C_+X, X) & \xrightarrow[f]{} & \pi_{i+1}(SX, C_-X) \end{array}$$

where the vertical isomorphisms come from the long exact sequence of pairs and f is induced by the inclusion $(C_+X, X) \hookrightarrow (SX, C_-X)$. Also, both C_+X and C_-X are $n-$connected if X is $(n-1)-$connected, and so Proposition 6.1 gives us that f is an isomorphism for $i + 1 < 2n$ and surjective for $i + 1 = 2n$.

□

6.2 Ring structure on stable homotopy groups

We will conclude our analysis of the homotopy groups of spheres by showing some remarkable, additional structure on the stable homotopy groups of spheres. This structure is that the stable homotopy groups, when considered as a direct sum $\pi_*^s = \bigoplus_{i \in \mathbb{N}} \pi_i^s$, π_*^s is a graded ring, and furthermore is graded commutative.

One good way of thinking of the stable homotopy groups π_i^s is as classifying maps of the form $S^{i+k} \to S^k$ for sufficiently large k. Then a natural multiplication operation comes as follows: Consider $i, j \in \mathbb{N}$, and let $k \in \mathbb{N}$ such that π_{i+j}^s classifies maps of the form $S^{i+j+k} \to S^k$. Then given $f \in \pi_i^s$ and $g \in \pi_j^s$ we can consider their product $fg \in \pi_{i+j}^s$, namely the map

$$S^{i+j+k} \xrightarrow{f} S^{j+k} \xrightarrow{g} S^k \tag{2}$$

which is in π_{i+j}^s for sufficiently large k. Looking initially, it seems plausible that $fg = (-1)^{ij} gf$ because the maps $S^{i+j+k} \xrightarrow{f} S^{j+k} \xrightarrow{g} S^k$ and $S^{i+j+k} \xrightarrow{g} S^{i+k} \xrightarrow{f} S^k$ are nearly identical.

It is immediately clear that this multiplication is well-defined on homotopy classes of maps, as any homotopy f_t or g_t extends immediately to a homotopy of the composition maps, $S^{i+j+k} \xrightarrow{f_t} S^{j+k} \xrightarrow{g_t} S^k$ and gives the desired homotopy between $fg = f_0 g_0$ and $f_1 g_1$. Associativity also follows trivially by associativity of composition of maps.

While distributivity ought to be checked, it follows immediately from commutativity and the straighforward fact that if $f, g : S^{i+j+k} \to S^{j+k}$ and $h : S^{j+k} \to S^k$, then $h(f+g) = hf + hg$ since both $h(f+g)$ and $hf + hg$ equal hf on one hemisphere of S^{i+j+k} and hg on the other.

So we only need to check graded commutativity, as follows:

Proposition 6.2. *The multiplication on π_*^s given in Equation 2 is graded commutative.*

Proof. Note first that we can consider the suspension of S^n, giving S^{n+1}, as the smash product $S^n \wedge S^1$, which is simply the suspension of S^n modulo the suspension of some point $x_0 \in S^n$. Thus under this identification, the suspension of a basepoint preserving map $f : S^n \to S^n$ becomes the smash product $f \wedge id_{S^1} : S^n \wedge S^1 \to S^n \wedge S^1$ and, by iterating suspensions, the k^{th} suspension of f, written $S^k f$ is $f \wedge id_S^K : S^n \wedge S^k \to S^n \wedge S^k$.

Now, let $f : S^{i+k} \to S^k$ and $g : S^{j+k} \to S^k$, where we have chosen k to be even. Then we have the following commutative diagram:

$$\begin{tikzcd}
S^{i+k} \wedge S^{j+k} \arrow[r, "f \wedge id"] & S^k \wedge S^{j+k} \arrow[r, "id \wedge g"] \arrow[d, "\sigma"'] & S^k \wedge S^k \arrow[d, "\tau"'] \\
& S^{j+k} \wedge S^k \arrow[r, "g \wedge id"] & S^k \wedge S^k
\end{tikzcd}$$

where σ and τ simply transpose factors. Then we can think of S^{j+k} and S^k as smash products of circles, giving that σ is composition of $k(j+k)$ transpositions of adjacent factors of circles. This has degree -1, and so σ has degree $(-1)^{k(j+k)}$, which is $+1$ since k is even.

Thus $\sigma \sim id$, and, by the same argument, $\tau \sim id$. This gives that $f \wedge g = (1 \wedge g)(f \wedge 1)$ is homotopic to the composition $(g \wedge 1)(1 \wedge f) \sim gf$. Similarly, $g \wedge f \sim fg$, so we only need to show that $f \wedge g \sim (-1)^{ij} g \wedge f$. To do this, consider the following commutative diagram:

$$\begin{tikzcd}
S^{j+k} \wedge S^{j+k} \arrow[r, "f \wedge g"'] \arrow[d, "\sigma"] & S^k \wedge S^k \arrow[d, "\sigma"] \\
S^{j+k} \arrow[r, "g \wedge f"'] & S^k \wedge S^k
\end{tikzcd}$$

where σ and τ are as before. Note that $\tau \sim id$, as before, but now σ has degree $(-1)^{(i+k)(j+k)}$. Since k is even, $(i+k)(j+k) \bmod 2 = ij \bmod 2$, and thus σ has degree $(-1)^{ij}$. Since additive inverses of homotopy groups are obtained by precomposing with a reflection of degree -1, we that $(g \wedge f) \circ \sigma$ is homotopic to $(-1)^{ij}(g \wedge f)$. Thus by commutativity of the diagram, $f \wedge g \sim (-1)^{ij} g \wedge f$, as π_*^s is graded commutative, as desired. □

7 Conclusion

In this document we have given a very brief introduction to the homotopy groups of spheres, including the stable homotopy groups. We have shown a few truly remarkable results, such as the suspension theorem (Theorem 6.1) and the cellular approximation theorem (Theorem 3.2) which, among other things, showcase the importance of CW-complexes in homotopy theory, and J.H.C. Whitehead's motivation for their definition.

Occasionally, throughout the exposition, we had to make a few assumptions (as in Lemmas 1.1 and 1.2, and the fact in the proof of Theorem 6.1). However, each time, these assumptions were minor, and played relatively minor rolls in the overall proof. We also directed the reader to a source where they could find a complete proof.

As mentioned in Section 5, there is a strong connection between homotopy and homology of a topological space. Unfortunately, we were only able to hint at the connection with Theorem 5.1. Given more time, and more space to work with, my next goal would be to thoroughly show the connection between (co)homology and homotopy theory, as given in Theorem 5.1 and Section 4.3 of [1].

References

[1] Hatcher, A. (2001) Algebraic Topology. New York, NY: Cambridge University Press.

[2] H. Toda, Note on the cohomology ring of certain spaces, *Proc. A.M.S.* 14 (1963), 89-95.