An Introduction to Characteristic Classes

Derek Sorensen

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When studying principal G-bundles, a useful tool for studying "how nontrivial" the bundles are (whatever that means) is to study its classifying map. If you look at the notes on principal G-bundles, we know that principal G-bundles over a CW-complex X are in bijective correspondence with the set $[X, BG]$ of homotopy classes of maps. However, in most cases it is not at all straightforward to find and study these maps. Instead, we use the ever-useful tool of cohomology to study the maps on cohomology which are induced by the maps in $[X, BG]$. This gives us the definition of a characteristic class of a principal G-bundle.

Definition 0.1. *Let $\pi : E \to B$ be a principal G-bundle with classifying map $[\varphi] \in [B, BG]$. If $c \in H^*(BG)$, then the characteristic class $c(E) \in H^*(B)$ is the image of c under the map $\varphi^* : H^*(BG) \to H^*(B)$.*

Immediately from this definition, we have that if E_1 and E_2 are isomorphic principal G-bundles over X, then their characteristic classes are isomorphic. Likewise, if $E = B \times G$ is the trivial principal G-bundle, then its classifying map is nullhomotopic, and so its characteristic classes must be trivial.

Theorem 0.2. *Given any fiber bundle $\pi : E \to B$ with fiber F and structure group $Aut(F)$, there exists a principal $Aut(F)$-bundle P such that $E = P \times_G F$.*

Characteristic classes can also be defined for arbitrary fiber bundles with fiber F and structure group $\mathrm{Aut}(F)$. Given such a bundle, we can show that there exists a principal $\mathrm{Aut}(F)$-bundle P such that $E = P \times_{\mathrm{Aut}(F)} F$ by using the following theorem:

Theorem 0.3. *Given any fiber bundle $\pi : E \to B$ with fiber F and structure group $Aut(F)$, there exists a principal $Aut(F)$-bundle P such that $E = P \times_G F$.*

We don't give a proof of this theorem here, but direct the reader to the author's notes on principal G-bundles.

Since P has a classifying map, we can study the characteristic classes of P via this classifying map. The definition of these characteristic classes is similar to that in Definition 0.1.

Definition 0.4. *Let $\pi : E \to B$ be a fiber bundle with fiber F, structure group $Aut(F)$ and associated principal $Aut(F)$-bundle P with classifying map $[\varphi] \in [B, BAut(F)]$. Then if $c \in H^*(BG)$, the characteristic class $c(E) \in H^*(B)$ is defined to be the image of c under the map $\varphi^* : H^*(BAut(F)) \to H^*(B)$.*

There are other formal ways to define characteristic classes, which defines each class as a functor that sends a vector bundle to its characteristic class in $H^*(B)$. These definitions are equivalent to those we have given (see [Kot12]).

We are now going to shift our attention to characteristic classes of vector bundles. As we noted in the notes on principal G-bundles, an n-dimensional real vector bundle is a fiber bundle with fiber $\mathbb{R}^n$ and, depending on whether or not we have inner products and orientations, structure group $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$ or $SO(n)$. For the purposes of this paper we will focus our attention on vector bundles with structure group $SO(n)$— i.e. on smooth, oriented vector bundles. A particularly useful example of such vector bundles are tangent bundles of smooth manifolds, which play a vital role in the material of this paper.

For a vector bundle $E \to B$, there are four main types of characteristic classes, which are:

1. Stiefel-Whitney classes $w_i(E) \in H^i(B; \mathbb{Z}/2)$ for a real vector bundle
2. Chern classes $c_i(E) \in H^{2i}(B; \mathbb{Z})$ for a complex vector bundle
3. Pontryagin classes $p_i(E) \in H^{4i}(B; \mathbb{Z})$ for a real vector bundle
4. The Euler class $e(E) \in H^n(B; \mathbb{Z})$ for an oriented n-dimensional vector bundle.

As it turns out, we can give a full description of $H^*(BSO(n); \mathbb{Z})$ in terms of the Potryagin and Euler classes. Our focus for the rest of this section will be to give a definition for the Pontryagin and Euler classes and then give the mentioned description. The Pontryagin classes are defined in terms of the Chern classes, which are elements of $H^{2*}(BSO(n))$ and which can in turn be defined by using Schubert cycles. We will not give a description of Chern classes, but direct the interested reader to [Hat09, Chapter 3]. The definition of Pontryagin classes is as follows:

Definition 0.5 (Pontryagin classes)**.** *Let $E \to B$ be a real vector bundle. Then the k^{th} (integral) Pontryagin class of E, written $p_k(E)$, is given by*

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(B; \mathbb{Z}),$$

where $E \otimes \mathbb{C}$ is the complexification of E, given by $E \otimes \mathbb{C} = E \oplus iE$.

The Euler is in some sense a refinement of the Pontryagin classes, as we will see later on. Recall first that the Euler class is for oriented vector bundles. Choosing an orientation for a vector bundle is the same as choosing a generator of $H^n(F, F \setminus F_0; \mathbb{Z})$ for each fiber, where F_0 is the zero element. Then via the Thom isomorphism (see, e.g. [Hat09, §3.2]) this gives us a so-called orientation class $u \in H^n(E, E \setminus E_0; \mathbb{Z})$, where E_0 is the zero section of E. Then using the zero section we get an inclusion $B \hookrightarrow E$, and thus inclusions of pairs

$$(B, \emptyset) \hookrightarrow (E, \emptyset) \hookrightarrow (E, E_0).$$

Then under the induced maps,

$$H^n(E, E \setminus E_0; \mathbb{Z}) \to H^n(E; \mathbb{Z}) \to H^n(B; \mathbb{Z})$$

the image of $u \in H^n(E, E \setminus E_0; \mathbb{Z})$ is the Euler class. It is a standard result that If $E \to B$ is an oriented vector bundle of dimension $2n$ then $e(E)^2 = p_n(E)$.

As we said before, the interests of this paper center on smooth, oriented vector bundles (tangent bundles of smooth manifolds) and their characteristic classes. In particular we are interested in the rational cohomology ring $H^*(BSO(n); \mathbb{Q})$, as $SO(n)$ is the structure group of such vector bundles. In a very convenient fashion, these rings have a very simple presentation.

Theorem 0.6. *$H^*(BSO(n); \mathbb{Q})$ is the $\mathbb{Q}$-polynomial ring generated by the Potryagin classes p_i and the Euler class e. If $n = 2d$, then*

$$H^*(BSO(2d); \mathbb{Q}) = \mathbb{Q}[p_1, ..., p_{d-1}, e].$$

Note that the generators of $H^*(BSO(2d); \mathbb{Q})$ do not include p_d because $e^2 = p_d$.

Characteristic classes of vector bundles are particularly well-understood and we now have excellent tools with which to work with them. This is not true for most other kinds of characteristic classes. In an arbitrary fiber bundle with fiber F and structure group $\mathrm{Aut}(F)$, the cohomology ring $H^*(B\mathrm{Aut}(F); G)$ can be very difficult to understand. In some situations, when the fiber bundle $E \to B$ has a smooth structure, one can define *generalized Miller-Morita-Mumford classes*, which are the simplest characteristic classes of smooth fiber bundles to understand. The reason they are so simple is because they draw on the characteristic classes of a natural, associated vector bundle over E and then push them down to classes in $H^*(B)$. That they are characteristic classes of the bundle $E \to B$ means that they are in the image of $H^*(B\mathrm{Diff}(F))$, where F is the fiber, and that they are natural with respect to bundle maps. Other than that, for a smooth manifold F, a priori the characteristic classes of a bundle with fiber F, given by $H^*(B\mathrm{Diff}(F))$, are not easy to understand. For a more thorough treatment of this subject, we direct the reader to [MS74] or [Kot12].

References

[Hat09] Allen Hatcher. Vector bundles and K-theory. Version 2.1, 5 2009.

[Kot12] Chris Kottke. Bundles, classifying spaces and characteristic classes. *Available at http://ckottke.ncf.edu/docs/bundles.pdf*, 5 2012.

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