(Co)Homology Theory

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This document contains the proofs of the main theorems of an introductory course in algebraic topology. They are intended for the student who is seeing this for the first time, and so the proofs are given in substantial detail, with (hopefully) intuitive explanations. This will take on a more algebraic approach, as this is what the author found most intuitive.

This is not intended to be a comprehensive treatment of simplicial and singular (co)homology theories, but instead a highlight of the main theorems which are important for the subject. This would be best used as a study guide, supplementary to Hatcher's *Algebraic Topology*, Massey's *Algebraic Topology: An Introduction*, or any other book on the subject that the reader finds useful.

Good luck!

Note: These notes are still under construction! I'll update the document as I write more. Suggestions and corrections are welcome.

1 Homology

Most of the theorems that we're going to include here prove that a certain tool can be used for calculating homology groups of complicated spaces using *exact sequences*. Exact sequences are to algebraic topology what integrals are to calculus. They are the principal tool for calculating homology groups, and they are extraordinarily powerful.

1.1 Mayer-Vietoris Exact Sequence

Trying to calculate the singular homology of a topological space is virtually intractable. The only space on which we can do that is the space with one point. (For the reader familiar with category theory, this is because, in the category of topological spaces **Top**, the space $\{*\}$ is terminal.) We need tools to calculate homology groups for more complicated spaces. One of the most powerful tools of these is the Mayer-Vietoris exact sequence.

The statement of the sequence is very straightforward, and the sequence becomes useful immediately in our quest to calculate homology groups of spaces that are not the one-point space.

Theorem 1. *Let $X = A \cup B$ be a topological space, where A, B are nonempty subsets of X such that X is the union of the interiors of A and B. If $A \cap B$ has a nonempty interior, there exists a long exact sequence of the form*

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to \cdots \to H_0(X) \to 0.$$

This result is actually very straightforward result of the so-called "Snake lemma." We ought to mention that if $\mathcal{A}$ and $\mathcal{B}$ are chain complexes, a *map of chain complexes* $\varphi : \mathcal{A} \to \mathcal{B}$ is a collection of maps $\varphi_i : A_i \to B_i$ for each i such that everything commutes, i.e. for every i, the following diagram is commutative:

$$\begin{CD} A_i @>{\varphi_i}>> B_i \\ @V{\partial_i^{\mathcal{A}}}VV @VV{\partial_i^{\mathcal{B}}}V \\ A_{i-1} @>{\varphi_{i-1}}>> B_{i-1}. \end{CD}$$

In the diagram above, the maps labeled $\partial_i^{\mathcal{A}}$ and $\partial_i^{\mathcal{B}}$ are the respective boundary maps of the chain complexes $\mathcal{A}$ and $\mathcal{B}$. What I claim here is that such a map induces a map of homologies. That is, we get a collection of maps $\phi_{i,*} : H_i(\mathcal{A}) \to H_i(\mathcal{B})$. This follows straight away from the fact that taking homology is a functor from the category of chain complexes to the category of abelian groups. If that doesn't make any sense to you, we'll prove it right here for good measure.

Proposition 1.1. *A map of chain complexes $\varphi : \mathcal{A} \to \mathcal{B}$ induces a map $\varphi_{i,*} : H_i(\mathcal{A}) \to H_i(\mathcal{B})$. Furthermore, the induced map of a composition is the composition of the induced maps, and the induced map of the identity map is the identity.*

Proof. We're going to prove this by defining a the map $\varphi_{i,*}$ using φ_i. This is done in the most naive way possible, namely, for any equivalence class $[a] \in H_i(\mathcal{A})$,

$$\varphi_{i,*}([a]) = [\varphi_i(a)].$$

Now we need to show that this map is well-defined by first showing that $\varphi_i(\ker(\partial_i^{\mathcal{A}})) \subset \ker(\partial_i^{\mathcal{B}})$ (i.e., that φ_i sends a cycle to a cycle), and then showing that if $a - b \in \operatorname{Im}(\partial_{i+1}^{\mathcal{A}})$ (i.e., $a - b$ is a boundary, meaning that $[a] = [b]$ under the homology quotient), then $\varphi_i(a) - \varphi_i(b)$ is a boundary in B_i. For convenience, we're going to use the vocabulary of cycles and boundaries. We're also going to leave out the $\mathcal{A}$ and $\mathcal{B}$ superscripts in $\partial_i^{\mathcal{A}}$ and $\partial_i^{\mathcal{B}}$, as it should be clear which one we mean each time we refer to the differential.

So first, suppose that $a \in A_i$ is a cycle. We want to show that $\varphi_i(a)$ is also a cycle. Since the diagram

$$\begin{CD} A_i @>{\varphi_i}>> B_i \\ @V{\partial_i}VV @VV{\partial_i}V \\ A_{i-1} @>{\varphi_{i-1}}>> B_{i-1} \end{CD}$$

commutes, and by definition $\partial(a) = 0$, it follows that $\partial \circ \varphi_i(a) = \varphi_{i-1} \circ \partial(a) = 0$, and $\varphi_i(a)$ is a cycle. So we're on the right track—now all we have to show is that $\varphi_{i,*}$ is well-defined on equivalence classes after the homology quotient.

To that end, suppose that $[a] = [b]$ in $H_i((A))$. This means that $a - b$ is a boundary in A_i. We want to show that $[\varphi_i(a)] = [\varphi_i(b)]$, which is the same as showing that $\varphi(a - b)$ is a boundary in B_i. This comes pretty quickly—Since $a - b$ is a boundary, there is some $c \in A_{i+1}$ such that $\partial_{i+1}(c) = a - b$. Since the diagram

$$\begin{array}{ccc} c \in A_{i+1} & \xrightarrow{\varphi_{i+1}} & B_{i+1} \\ {\scriptstyle \partial_{i+1}}\downarrow & & \downarrow{\scriptstyle \partial_{i+1}} \\ a - b \in A_i & \xrightarrow{\varphi_i} & B_i \end{array}$$

commutes, we know that $\partial \circ \varphi_{i+1}(c) = \varphi_i(a-b)$, and so $\varphi_i(a-b)$ has a preimage under ∂_{i+1}. Thus $\varphi_i(a-b)$ is a boundary, as desired.

Now to show that the induced map of a composition of maps is the composition of the induced maps, just replicate the arguments above, but try to extend it for commuting diagrams that look like this:

$$\begin{array}{ccccc} A_{i+1} & \xrightarrow{\varphi_{i+1}} & B_{i+1} & \xrightarrow{\psi_{i+1}} & C_{i+1} \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ A_i & \xrightarrow{\varphi_i} & B_i & \xrightarrow{\psi_i} & C_i \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ A_{i-1} & \xrightarrow{\varphi_{i-1}} & B_{i-1} & \xrightarrow{\psi_{i-1}} & C_{i-1}. \end{array}$$

One ought to take care to make sure this diagram commutes, but it should be readily visible. And finally, the fact that the identity induces the identity should also be readily visible. □

OK, now that we've established that a map of chain complexes descends to a map of homologies for each degree, we can take a look at the Snake lemma. We ought to note here that this lemma is standard, and that anyone who studies algebraic topology or homological algebra should know this (not too difficult) proof.

Lemma 1.1 (The Snake Lemma.)**.** *Consider a short exact sequence of chain complexes*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0.$$

Then for each $i > 0$ there exists a map

$$H_i(\mathcal{C}) \to H_{i-1}(\mathcal{A}).$$

These maps, along with the induced maps on homology, give us a long exact sequence of the form

$$\cdots \to H_i(\mathcal{A}) \to H_i(\mathcal{B}) \to H_i(\mathcal{C}) \to H_{i-1}(\mathcal{A}) \to \cdots \to H_0(\mathcal{C})$$

Proof. Let's draw the full diagram corresponding to these maps of chain complexes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{i+1} & \longrightarrow & B_{i+1} & \longrightarrow & C_{i+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\ 0 & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\ 0 & \longrightarrow & A_{i-1} & \longrightarrow & B_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & 0. \end{array}$$

The main challenge is that we need to find a way to map the kernel of ∂ in C_i to the kernel of ∂ in A_{i-1} (again, these two instances of ∂ refer to different maps). To find the map, take an element $c \in \ker(\partial) \subseteq C_i$. Note that the rows of the diagram are exact (that's what it means for the sequence of chain complexes to be exact). So we can find a preimage of c, call it b, in B_i. Hitting b by ∂, we get $b' \in B_{i-1}$. The diagram commutes, and so the horizontal image of b' is 0. By exactness, we have a preimage of b' in A_{i-1}. Call that a, and define the map of homology as

$$[c] \mapsto [a].$$

Of course, we're not done. We have to prove (1) that this map is well-defined. We then have to prove exactness at each point in what we claimed in the lemma to be the long exact sequence. This is an excellent exercise in diagram chasing. If the reader can't readily verify that these facts, she may need more practice in diagram chasing and, considering how much easier it is to diagram chase on your own rather than reading it from some proof, I would be depriving the reader of a much-needed exercise by providing the details here. If it is not hard for the reader to verify the rest of this proof, then she doesn't need to read the proof here anyway. □

Great. Now that we have the Snake lemma established, all we need to do to prove the Mayer-Vietoris sequence is to provide a suitable short exact sequence of chain complexes that will give us what we want. (Before reading further, by looking at the Mayer-Vietoris sequence, can you guess what it is?) We will write $\mathcal{C}(X)$ to be the singular chain complex of a given space X. The sequence we want is the sequence of chain maps made up of maps of the form:

$$0 \to \mathcal{C}_i(A \cap B) \xrightarrow{+} \mathcal{C}_i(A) \oplus \mathcal{C}_i(B) \xrightarrow{-} \mathcal{C}_i(X) \to 0. \tag{1}$$

Here, the first nontrivial map, labeled by $+$, is the diagonal map (i.e. it sends $\delta \mapsto (\delta, \delta)$), and the map labeled $-$ is the natural subtraction map, which sends $(\delta, \gamma) \mapsto \delta - \gamma$. Note that the $+$ map is injective (clearly), and that $-$ is surjective.

The kernel of $-$ consists of elements (δ, γ) such that $\delta - \gamma = 0$ in $\mathcal{C}_i(X)$. Since $\mathcal{C}(X)$ is a free abeliean group, $\delta - \gamma = 0$ iff $\delta = \gamma$. This can only happen if $\delta = \gamma$ is in both A and B, and thus in their intersection, and we have that $\delta \in \mathcal{C}_i(A \cap B)$. So (1) from above is indeed exact, just like we wanted. And the fact that this short exact sequence gives rise to the long exact sequence that we wanted in the first place comes by simply taking the homology of each group. What do you get?

1.2 Excision

Excision sounds scary, but it's no big deal. There are two versions of the theorem. We will prove that they are equivalent.

Excision is also extremely important. It doesn't hold for homotopy groups, which is one of the things that makes them so difficult to calculate. We don't even know homotopy groups for most spheres!

1.3 Equivalence of Simplicial and Singular Homologies

1.4 Simplicial Approximation

1.5 Homology and the Fundamental Group

2 Cohomology

2.1 Ring Structure and the Cup Product

2.2 Universal Coefficient Theorem

2.3 Künneth Formula

2.4 Poincaré Duality

3 Additional Topics

3.1 Local Coefficients