An Introduction to Principal G-Bundles

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These notes build on the notes of vector and fiber bundles. An important example of a fiber bundle is a principal G-bundle. Indeed, it is on the theory of principal G-bundles that the theory of characteristic classes, and thus of this entire document, rests. In this section we will give a brief introduction to principal G-bundles, following the notes Mitchell and Kottke in, repsectively [Mit01] and [Kot12]. We recommend that the interested reader consult these notes for a more in-depth treatment of the subject.

Let G be a topological group. Then a *left G-space* is a topological space X equipped with a continuous left G-action $G \times X \to X$. Equivalently, a left G-space is a space X equipped with a group homomorphism from G to the group of homeomorphisms $X \to X$. If X and Y are G-spaces, then a *G-equivariant map* is a map $\phi : X \to Y$ such that $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X$.

Now let E and B be G-spaces such that the action of G on B is trivial, and consider a G-map $\pi : E \to B$. Then $\pi : E \to B$ is a *principal G-bundle* if it satisfies similar local triviality conditions as a fiber bundle. That is, B has an open cover $\{U_\alpha\}_{\alpha \in I}$ such that, for all α there exist G-equivariant homeomorpihsms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U \times G$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U \times G \\ \big\downarrow{\scriptstyle\pi} & \swarrow & \\ U. & & \end{array}$$

Note that the fibers are copies of G. The G-equivariant homeomorphisms ϕ_α could be any map which makes the diagram commute, and so in particular there is not always a canonical identity element in $\pi^{-1}(b)$ for any particular b. We call these fibers *G-torsors*, which are to groups what affine spaces are to vector spaces.

Example 0.1. *A normal covering map (i.e. a covering map corresponding to a normal subgroup of the fundamental group of the base space) is a principal G-bundle, where G*

is the group of deck transformations.

Note that the G-equivariant homeomorphisms ϕ_α give us a canonical G-action on $\pi^{-1}(U_\alpha)$, given by $g \cdot (u, h) = (u, g \cdot h)$. Furthermore, this action is free and transitive. Thus B is the orbit space of the G-space E, i.e. $B \cong E/G$. We proceed with a basic fact about principal G-bundles. For proofs of the results in this section which we do not supply, we direct the reader to [Mit01].

Lemma 0.2. *Any morphism of principal G-bundles is an isomorphism.*

Now let $\pi : P \to B$ be a principal G-bundle and consider a map $f : B' \to B$. We allow this to be any continuous map, and then give it the structure of a G-equivariant map simply by endowing B' with the structure of a G-space via the trivial G-action. We can form the category theoretic pullback $P' \equiv f^*P \equiv B' \times_B P$; it is easy to see that P' inherits the structure of a principal G-bundle over B' from P.

We can immediately note that, as a purely categorical fact, bundle maps $Q \to P'$ are in bijective correspondence with commutative diagrams of the form:

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B. \end{array} \tag{1}$$

By Lemma 0.2 we have that Q is isomorphic to P' if and only if there exists a commuting diagram such as (1). Thus, for any given map $B' \to B$ there exists only one possible principal G-bundle, up to isomorphism, which will make (1) commute. The following fact, Theorem 0.3, allows us to go further and say that for any given homotopy class of maps in $[B', B]$ there exists a principal G-bundle unique up to isomorphism which makes (1) commute.

Theorem 0.3. *Let $P \to B$ be a principal G-bundle over an arbitrary space B, and suppose that X is a CW-complex. Then if $f, g : X \to B$ are homotopic maps, the pullbacks f^*P and g^*P are isomorphic as principal G-bundles over X.*

If we are given a principal G-bundle $P \to B$ and a CW-complex B', we could in theory classify those principal G-bundles which can be achieved as pullbacks of $P \to B$ by the homotopy class of maps which pull it back. A priori, we might run into problems because assigning a principal G-bundle the map by which it is a pull-back of $P \to B$ might not even be well-defined. We would like for this notion to be well-defined, and ideally to be able to say that *any* principal G-bundle over B' is a pullback of $P \to B$. As it turns out, this can be achieved for certain types of principal G-bundles.

Theorem 0.4. *Let $P \to B$ be a principal G-bundle. Then P is weakly contractible if and only if for all CW-complexes X, there is a bijective correspondence between $[X, B]$ and principal G-bundles over X via the map $f \mapsto f^*P$.*

We can actually relax the condition that X be a CW-complex to just requiring paracompactness, but for our purposes we will simply use CW-complexes. Under the hypotheses of Theorem 0.4 hypotheses we call B a *classifying space* of G and $P \to B$ a *universal bundle*. A classifying space of a topological group G is most commonly written as BG, while the universal bundle is most commonly written EG. As it turns out, classifying spaces and universal bundles of a topological group are unique up to homotopy equivalence, and so BG and EG are often referred to as "the" classifying space (resp. universal bundle) of G when only the homotopy type is needed. In particular, when investigating the homology, cohomology or homotopy groups of a classifying space we can refer to "the" classifying space BG. We give another remarkable result, which will allow us, among other things, to define characteristic classes in the next section.

Theorem 0.5. *Let G be a topological group. There exists a classifying space for G.*

We will finish this section by giving the theory of balanced products and structure groups, which will also play an important role in the definition of characteristic classes. Let W be a right G-space and X a left G-space. Then the *balanced product* $W \times_G X$ is the quotient space $W \times X/\sim$, where $(wg, x) \sim (w, gx)$. (We note here that this is different from a pullback, despite the similarity in notation.) We could equivalently convert X into a right G-space by setting $gx = xg^{-1}$ and take the orbit space of $W \times X$ under the diagonal action $(w, x)g = (wg, g^{-1}x)$. Note that if $X = *$ is a point, then $W \times_G *$ is simply the orbit space W/G.

Now, suppose that $\pi : E \to B$ is a principal G-bundle and let F be a left G-space. Since $F \to *$ is G-equivariant, and $E \times_G * = B$ we have an induced map $E \times_G F \to E \times_G * = B$ which has the structure of a fiber bundle with fiber F. We call a local product of this form a *fiber bundle with fiber F and structure group G*. We also call $E \times_G F$ the *associated fiber bundle* to E with fiber space F. Because F is a left G-space, there is a group homomorphism $G \to \mathrm{Aut}(F)$ corresponding to the left action. In most of our examples we will be interested in the case that $G = \mathrm{Aut}(F)$ and this homomorphism is a group isomorphism.

Example 0.6. *An n-dimensional real vector bundle is a fiber bundle with fiber $\mathbb{R}^n$ and structure group $GL_n(\mathbb{R})$. If we give our vector bundle an inner product, then* *the structure group will be $O(n)$. If we give our vector bundle an orientation, then the structure group will be $SL_n(\mathbb{R})$ or $SO(n)$. The analogous results hold for complex vector bundles.*

The final result of this section forms the basis for the theory of our next section, characteristic classes.

Theorem 0.7. *Given any fiber bundle $\pi : E \to B$ with fiber F and structure group $Aut(F)$, there exists a principal $Aut(F)$-bundle P such that $E = P \times_G F$.*

In light of Theorem 0.7, consider a fiber bundle $\pi : E \to B$ with fiber F. We can choose $\mathrm{Aut}(F)$ depending on the bundle we're interested—for example, if we are looking at a smooth bundle then we let $\mathrm{Aut}(F) = \mathrm{Diff}(F)$, the diffeomorphism group of F. We can then find its associated principal $\mathrm{Aut}(F)$-bundle (i.e. the bundle P such that $E = P \times_{\mathrm{Aut}(F)} F$). By Theorem 0.4 there is a homotopy class of maps, called the classifying map, which classifies P as a principal $\mathrm{Aut}(F)$-bundle. It is these maps upon which the theory of characteristic classes is built.

References

[Kot12] Chris Kottke. Bundles, classifying spaces and characteristic classes. *Available at http://ckottke.ncf.edu/docs/bundles.pdf*, 5 2012.

[Mit01] Stephen A. Mitchell. Notes on principal bundles and classifying spaces. *Available at https://www3.nd.edu/ mbehren1/18.906/prin.pdf*, 8 2001.