# On the Tautological Ring of $g(S^d \times S^d)$

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*Dominus illuminatio mea*

*The Lord is my light*

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## Abstract

The *tautological ring* of a manifold $M$, denoted $R^*(M)$, is a ring generated by *generalized Mumford-Miller-Morita classes*, which are characteristic classes of smooth fiber bundles with fiber $M$. These characteristic classes are defined via the smooth structure on the bundle. We study the tautological ring of $g(S^d \times S^d)$ modulo the nilradical $\sqrt{0}$. Our first objective is to give the proof of [GGRW17, Theorem 1.1], which gives a complete description of $R^*(g(S^d \times S^d))/\sqrt{0}$ when $d$ is odd. We then give insights into the possibility of proving a similar result for the case that $d$ is even. Our main contribution to the problem comes in Section 6, where we prove that a crucial supporting theorem is false when $d$ is even, showing that the methods used in the proof of [GGRW17, Theorem 1.1] cannot be used in this case. We finish by proposing some strategies for moving forward to prove an analogous result to [GGRW17, Theorem 1.1] in the case that $d$ is even.

# Contents

# 1 Introduction

Consider a smooth manifold $M$. The *tautological ring* of $M$, written $R^*(M)$, is a ring generated by *generalized Mumford-Miller-Morita classes.* These are characteristic classes of smooth fiber bundles with fiber $M$, defined via the smooth structure on the bundle (see §1.1). Tautological rings are known for some cases, such as when $M = S^n$. However, there are still many basic examples of manifolds for which the tautological ring remains unknown. As noted in [RW16], the simplest of these is $\mathbb{C}P^2$. The connected sum of $g$ copies of $S^d \times S^d$, which we denote simply as $g(S^d \times S^d)$, is also one of these spaces. It is to this class of spaces which we will devote our attention.

In an effort to understand the tautological ring $R^*(g(S^d \times S^d))$, we will study the quotient ring $R^*(g(S^d \times S^d))/\sqrt{0}$, where $\sqrt{0}$ denotes the nilradical. We will devote most of our attention to proving the main result of [GGRW17], which gives a complete description of $R^*(g(S^d \times S^d))/\sqrt{0}$ when $d$ is odd and for all $g$. As we will see, many of the results in [GGRW17] can be applied to the case that $d$ is even, but not sufficiently to give a complete description. As we proceed through the proof, we will make notes when a particular result does not apply to the $d$ even case and show the reasons for this. The arguments for much of this proof rely heavily on the main technical result of [Gri13] which, if it could be generalized to the $d$ even case, would make a substantial contribution to understanding $R^*(g(S^d \times S^d))/\sqrt{0}$ if $d$ is even. Our contribution to this problem is showing, in Section 6, that this result is false if $d$ is not odd, and thus cannot be generalized in this way. We cannot yet give any conclusive results about the structure of $R^*(g(S^n \times S^n))/\sqrt{0}$ when $n$ is even; we will, however, give some insights and ideas into what might be done to continue research on this topic.

Working with characteristic classes of manifolds requires the understanding of several preliminary subjects. Here at the beginning we will only cover some of the essential definitions and background which we need, assuming the reader is familiar with fiber bundles, fibrations, principal $G-$bundles, characteristic classes, cohomology with local coefficients and the Serre spectral sequence. For the reader who is unfamiliar with any of these subjects, we have provided a brief introduction to each of them in the Appendix, as well as references from which the reader can read further.

We will proceed as follows: We first give some preliminary definitions in Section 1.1. In Section 2, we will introduce the main result of [GGRW17], given here as Theorem 2.1, and give a proof relying on two results given in the same paper. We will then focus on one of these results, Proposition 2.2, and its proof. Sections 4 and 5 will be devoted to various theorems and lemmas which constitute the proof of Proposition 2.2.

In Section 6 we show that one of the primary results used in the $d$ odd case is false when $d$ is even, and in Section 7 we discuss the possible avenues for research in the future. As promised, an appendix follows. Our goal is to give sufficient background material and ideas for proceeding with this problem that any reader could begin research after having thoroughly studied this document.

## 1.1 Preliminary Definitions

We will begin by defining our main object of study: the tautological ring of a smooth manifold $M$, denoted $R^*(M)$. As we noted in the introduction, tautological rings are generated by certain characteristic classes of smooth manifold bundles, called *generalized Miller-Morita-Mumford classes*, or *kappa classes.* In particular, for the definition of $R^*(M)$ we are interested in the kappa classes of smooth fiber bundles with fiber $M$.

Thus suppose that $M$ is a $d$-dimensional smooth manifold, and that $E^{k+d}$ and $B^k$ are connected compact smooth oriented manifold bundles. Let $\pi : E \to B$ be a smooth fiber bundle with fiber $M^d$. Inherent to the smooth structure of $\pi$ is a vector bundle over $E$, called the *vertical tangent bundle*, defined as $T_\pi := \ker(D\pi)$ (see Example A.10). The characteristic classes of $T_\pi$ are given by $H^*(BSO(d); \mathbb{Q})$.

Then taking an element $c \in H^*(BSO(d); \mathbb{Q})$, we define the *generalized Miller-Morita-Mumford class* (or *kappa class*) by use of a map $\pi_! : H^*(E; \mathbb{Q}) \to H^{*-d}(B; \mathbb{Q})$ called the *pushforward map.* In the context of our work there are two equivalent ways to define this map. The first, as is done in [GGRW17] and [RW16], is by use of a fiber integral. For each $c \in H^*(BSO(d); \mathbb{Q})$, we can get a class in $H^{*-d}(B; \mathbb{Q})$ given by

$$\kappa_c(\pi) = \int_\pi c(T_\pi) \in H^{*-d}(B; \mathbb{Q}).$$

Using this, $\pi_!$ is given by $c \mapsto \kappa_c(\pi)$. The other definition, as given in [Gri13], is more algebraic and makes use of the Serre spectral sequence. The definition of $\pi_!$ using this method is rather lengthy, so we will say no more here and fully address the definition in Section 4.

These are the simplest characteristic classes of bundles with fiber $M$ and structure group Diff$M$, owing to the fact that we can take full advantage of the knowledge of vector bundles instead of trying to understand a complete description of $H^*(B\mathrm{Diff}(M))$ for an arbitrary smooth manifold $M$ [RW16]. In particular, it is well known that $H^*(BSO(d); \mathbb{Q})$ is generated by the Pontryagin and Euler classes. In the case that $M$ has even dimension $2d$, $H^*(BSO(2d); \mathbb{Q}) = \mathbb{Q}[p_1, p_2, ..., p_{d-1}, e]$, so, for example, we can form the kappa classes $\kappa_{p_1}(\pi), \kappa_{p_2}(\pi), ..., \kappa_{p_{d-1}}(\pi), \kappa_e(\pi)$ or, as we will use later, the classes $\kappa_{ep_1}(\pi), \kappa_{ep_2}(\pi), ..., \kappa_{ep_{d-1}}(\pi)$, and so on.

To define the tautological ring, we consider a basis $\mathcal{B} \subset H^*(BSO(d);\mathbb{Q})$. Note that by using $\pi$ we can define a ring homomorphism

$$\gamma(\pi) : \mathbb{Q}[\kappa_c \mid c \in \mathcal{B}] \to H^*(B;\mathbb{Q})$$

given by

$$\kappa_c \mapsto \kappa_c(\pi).$$

We let $I_M$ be the ideal of $\mathbb{Q}[\kappa_c \mid c \in \mathcal{B}]$ generated by the polynomials in the $\kappa_c$ which vanish under $\gamma(\pi)$ for *all* smooth fiber bundles $\pi$ with fiber $M$. The *tautological ring* is the quotient ring

$$R^*(M) := \mathbb{Q}[\kappa_c \mid c \in \mathcal{B}]/I_M.$$

The structure of $I_M$, and thus $R^*(M)$, depends on the structure of *all* fiber bundles that can be formed with fiber $M$, and if $\kappa_c \in I_M$ then on any given fiber bundle it either vanishes because $c(T_\pi) = 0$ or because $c(T_\pi) \neq 0$ and $c(T_\pi) \in \ker \pi_!$.

Let us consider $R^*(M)$ in the case that $M = g(S^d \times S^d)$. Let $\pi : E \to B$ be some smooth fiber bundle with fiber $M$. Since $\dim(M) = 2d$, the characteristic classes of $T_\pi$ are given by $H^*(BSO(2d);\mathbb{Q}) = \mathbb{Q}[p_1, p_2, ..., p_{d-1}, e]$, where the $p_i$ are the Pontryagin classes and $e$ is the Euler class. This gives us a natural basis $\mathcal{B} = \{p_1, ..., p_{d-1}, e\}$, and a ring homomorphism

$$\gamma(\pi) : \mathbb{Q}[\kappa_{p_1}, \kappa p_2, ..., \kappa_{p_{d-1}}, \kappa_e] \to H^*(B;\mathbb{Q}).$$

Of course, to define $R^*(M)$ we need $I_M$, which can be defined as the intersection of $\ker(\gamma(\pi))$, taken over all smooth fibre bundles $\pi$, and get $R^*(M)$ by taking the appropriate quotient ring. A different and useful way of thinking of $R^*(M)$, as stated in [RW16], is as the subring

$$R^*(g(S^d \times S^d)) \subset H^*(B\mathrm{Diff}^+(M);\mathbb{Q})$$

generated by the $\kappa_{p_i}$ and $\kappa_e$.

Thus the trick to finding the tautological ring of a manifold is finding the polynomials in elements of $\mathcal{B}$ which vanish under $\gamma(\pi)$ for all smooth fiber bundles $\pi$. As we have already mentioned, we will not be so ambitious as to give a full description of $R^*(g(S^n \times S^n))$, but instead will investigate the tautological ring modulo the nilradical, or the ideal of nilpotent elements, which is denoted $\sqrt{0}$. This makes the problem more tractable, for instead of showing that a polynomial $p$ is in the kernel of $\gamma(\pi)$ for all smooth fiber bundles $\pi$, we only need to show that $\gamma(\pi)(p)$ is *nilpotent* for all $\pi$, which is a strictly weaker condition. As the reader will see, nilpotence plays a fundamental role in most of the proofs we will present.

## 2 Main Theorem

We will now introduce the main known result on the structure of $R^*(g(S^d \times S^d))/\sqrt{0}$, which is [GGRW17, Theorem1.1]. Our focus will be to explore the parts of the proof which provide insight to the nature of the work yet to be done in this area. Because of that, we will be unable to give a full proof of this theorem, but will instead focus on one of the main, supporting results and the theory which supports it. We will write $W_i := i(S^d \times S^d)$, using the convention that $W_0 = S^{2d}$.

**Theorem 2.1** (Main result)**.** *Let $d$ be odd. Then*

1. $R^*(W_0)/\sqrt{0} = \mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_d}]$
2. $R^*(W_1)/\sqrt{0} = \mathbb{Q}$,
3. $R^*(W_g)/\sqrt{0} = \mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_{d-1}}]$

As stated in [GGRW17, §5.3], $(i)$ holds before taking the quotient by the nilradical, and also holds for both even and odd.

The most obvious restriction on the scope of this theorem is the requirement that $d$ be odd. It is on this detail that we will focus our attention. The need for this hypothesis is complicated and nuanced, and it will take the bulk of this document to give a full description. Theorem 2.1 relies on two results and follows very simple logic. In the first result, we find generators for $R^*(W_g)/\sqrt{0}$, and in the second result we show their algebraic independence. We include the two results, [GGRW17, Corollary 3.4] and [GGRW17, Theorem 4.1], as follows:

**Proposition 2.2** (Generators)**.** *Let $d$ be odd.*

*(i)* $R^*(W_0)/\sqrt{0}$ *is generated by* $\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_d}$,

*(ii)* $R^*(W_1)/\sqrt{0} = \mathbb{Q}$,

*(iii)* $R^*(W_g)/\sqrt{0}$ *is generated by* $\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_{d-1}}$.

Note that the statement of (ii) simply says that every kappa class is nilpotent, and the only difference between $R^*(W_g)/\sqrt{0}$ and $R^*(W_0)/\sqrt{0}$ is that $\kappa_{ep_{d-1}}$ is nilpotent in $R^*(W_g)/\sqrt{0}$.

**Proposition 2.3** (Algebraic independence)**.** *Let $d$ be either odd or even, and $\varepsilon = 1$ if $d$ is odd. Then*

*(i) the map $\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_d}] \to R^*(W_0)/\sqrt{0}$ is injective,*

*(ii) the map $\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_{d-\varepsilon}}] \to R^*(W_g)/\sqrt{0}$ is injective for $g > 1$.*

The reader may have noticed that the hypotheses of Proposition 2.3 explicitly state that $d$ can be either odd or even. Since we have the injections from $(i)$ and $(ii)$, we know that in the case that $d$ is even $R^*(W_g)/\sqrt{0}$ is *at least* as big as the case that $d$ is odd—indeed when $d$ is even, by $(ii)$ we know that if $g > 1$, $R^*(W_g)/\sqrt{0}$ has strictly more generators than it does when $d$ is odd. The next step for the $d$-even case is thus to either show that the injections in $(i)$ and $(ii)$ are also surjections (which is done for the case that $d$ is odd in Proposition 2.2), or show that these are insufficient and find a strictly larger set of generators.

We might naively hope that the arguments used to prove Proposition 2.2 could somehow be altered only slightly to do this. Unfortunately, this is not the case. As we will see, the proof of Proposition 2.2 relies fundamentally on the parity of $d$.

## 3 Generators

This and the next two sections will be devoted to addressing the proof of Proposition 2.2, the result which gives the generators of $R^*(W_g)/\sqrt{0}$ for all $g$. As we did for the main theorem in Section 2, we will first give a straightforward proof, relying on results which we state from [GGRW17]. In Sections 4 and 5 we will prove the supporting results.

Our first supporting result comes from [Gri13], and gives us a very useful relationship between elements of even degree and the image of their cup product under $\pi_!$. The result is that if $d$ is odd, $a, b \in \ker \pi_!$ have even degree and $\pi : E \to B$ is an oriented manifold bundle with fiber $W_g$, then some power of $\pi_!(a \cup b)$ is torsion. Note that with rational coefficients, the $\pi_!(a \cup b)$ is *nilpotent*, which means it is zero after the quotient by the nilradical.

We emphasize here that $\pi_!$ is a group homomorphism on the underlying abelian groups of $H^*(E; \mathbb{Q})$ and $H^{*-2d}(B; \mathbb{Q})$, but is *not* a ring homomorphism, as $\pi_!(a \cup b)$ need not vanish. That $\pi_!$ is a group homomorphism with respect to the group operation of addition (and that the map does not respect multiplication) is evident from its definition by use of a fiber integral. This fact will also become clear in Section 4 from the more algebraic definition. We include here Grigoriev's result.

**Theorem 3.1** (Theorem 2.7 in [Gri13])**.** *Let $d$ be an odd natural number. Let $\pi : E \to B$ be an oriented manifold bundle with fiber $W_g$ and let $a, b \in H^*(E;\mathbb{Z})$ be two classes such that $\pi_!(a) = 0$, $\pi_!(b) = 0$, and $\deg(a)$ is even.*

*Then the classes $\pi_!(a \cup a) \in H^{2\deg(a)-2d}(B;\mathbb{Z})$ and $\pi_!(a \cup b) \in H^{\deg(a)+\deg(b)-2d}(B;\mathbb{Z})$ satisfy the following two relations.*

$$(2g+1)! \cdot \pi_!(a \cup a)^{g+1} = 0 \tag{1}$$

$$(2g+1)! \cdot \pi_!(a \cup b)^{2g+1} = 0 \tag{2}$$

Note the difference in exponents between Equations (1) and (2). We also note that instead of requiring that the fiber be $W_g$, we only need the fiber to be $(d-1)$-connected and have top cohomology isomorphic to $\mathbb{Z}$ (which Grigoriev calls a *highly connected manifold of dimension* $2d$). The reason for this will become obvious when we give the proof of Theorem 3.1 in Section 4. The proof of Proposition 2.2 does not use Theorem 3.1 as is, but instead uses the following, stronger result which can be formally deduced from Theorem 3.1.

**Theorem 3.2** (Theorem 3.1 in [GGRW17])**.** *Let $\pi : E \to B$ be a fibration with homotopy fiber homotopy equivalent to $W_g = g(S^d \times S^d)$, for $d$ odd, such that the action of $\pi_1(B, b)$ on $H^{2d}(\pi^{-1}(b);\mathbb{Q})$ is trivial. For classes $a, b \in H^*(E;\mathbb{Q})$ such that $A = |\pi_!(a)|$ and $B = |\pi_!(b)|$ are even, if $\pi_!(a)^k = 0$ and $\pi_!(b)^l = 0$, then $\pi_!(a \cup b)^{(2g+1)(Ak+Bl)} = 0$.*

We will not prove Theorem 3.2, except to say that in the proof we use Theorem 3.1 to conclude, after pulling back the classes $a$ and $b$ to classes $a'$ and $b'$ which are in the kernel of $\pi_!$, that $\pi_!(a' \cup b')$ is nilpotent. The curious reader can read the full proof from [GGRW17, Theorem 3.1]. We move on to the second result which we use for the proof of Proposition 2.2.

**Proposition 3.3** (Proposition 3.3 in [GGRW17])**.** *Let $d$ be odd, $I = (i_1, ..., i_d)$ be a sequence, $p_I = p_1^{i_1} p_2^{i_2} \cdots p_n^{i_d}$ be the associated monomial in the Pontrjagin classes, and write $|I| = \sum_{j=1}^{d} i_j$. Then:*

*(i) The class $\kappa_{p_I}$ is nilpotent in $R^*(W_g)$.*

*(ii) We have*

$$\chi^{|I|} \cdot \kappa_{ep_I} = \prod_{j=1}^{d} \kappa_{ep_j}^{i_j} \in R^*(W_g)/\sqrt{0} \tag{3}$$

*(iii) If $g \geq 1$ then for all $k > 1$ the classes $\kappa_{e^k}$ is nilpotent in $R^*(W_g)$.*

We stop here to make the comment that the proof of this result relies on the fact that the kappa classes induced by a specific type of characteristic classes are nilpotent in $R^*(W_g)$. These are called the modified Hirzebruch $\mathcal{L}$-classes and denoted $\tilde{\mathcal{L}}_i \in H^{4i}(BSO(2d);\mathbb{Q})$. One of the results upon which the proof of Proposition 3.3 relies is the following.

**Theorem 3.4** (Theorem 2.1 in [GGRW17])**.** *Let $M$ be a manifold of dimension $2d$. The classes $\kappa_{\tilde{\mathcal{L}}_i} \in R^*(M)$ are nilpotent for all natural numbers $i \geq 1$ such that $4i - 2d \neq 0$.*

Under the hypothesis that $d$ is odd, Theorem 3.4 gives us that all the kappa classes of the form $\kappa_{\tilde{\mathcal{L}}_i}$ are nilpotent in $R^*(M)$. However, if $d$ is even, we do not have this result for $i = d/2$. In Section 6 we will explore the nature of the relationship that this theorem has to the overall theory, and what obstructions it contributes to extending to the $d$-even case.

Finally, we give the proof of Proposition 2.2.

*Proof of Proposition 2.2.* Proposition 3.3 (i) and (ii) give immediately that for any $g$, the ring $R^*(W_g)/\sqrt{0}$ is generated by the elements $\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_d}$.

In the case that $g = 1$, we have that $\kappa_e = 0$ (see [GGRW17, Corollary 3.4]). It follows from Theorem 3.2 and Proposition 3.3 (i) that each of the $\kappa_{ep_i}$ are nilpotent, and so $R^*(W_1)/\sqrt{0} = \mathbb{Q}$.

In the case that $g > 1$, Proposition 3.3 (iii) implies that $\kappa_{ep_d = \kappa_{e^3}}$ is nilpotent. So $R^*(W_g)/\sqrt{0}$ is generated by the elements $\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_{d-1}}$. □

## 4 Theorem 3.1 and the $d$−odd hypothesis

Our goal for the next two sections will be to investigate the precise reasons for which we need the hypothesis that $d$ be odd in Proposition 2.2. Later, in Sections 6 and 7, we will explore the possibility of extending to the case that $d$ is even.

We point out that both of the supporting results in the proof of Proposition 2.2 use the hypothesis that $d$ is odd. Thus in order to understand fully the need for the $d$-odd hypothesis, we will need to investigate the proof of both supporting results to find the precise reasons for which the hypothesis is used. We will begin with Theorem 3.2 in this section, and then in Section 5 we will investigate Proposition 3.3.

The authors note in [GGRW17, p.2], "We cannot obtain results as conclusive as Theorem 2.1 for $d$ even, as our argument relies on [Gri13] which does not apply in this case." What is referred to here comes out in this paper in Section 3, where we used Theorem 3.1, given in [Gri13], in order to prove Theorem 3.2. Thus showing why Theorem 3.2 depends on the $d$-odd hypothesis comes down to showing why Theorem 3.1 depends on it. We will thoroughly give the proof of Theorem 3.1, emphasizing the points at which we use the hypothesis and pointing out why it is not possible to use the same arguments for the case that $d$ is even. The proof we give will follow the arguments given in [Gri13].

This section will draw heavily on the material from Appendices D and E. In particular, we will:

- use the system of local coefficients denoted by $\mathcal{H}^i$ and corresponding to the cohomology groups $H^i(M;\mathbb{Z})$, where $M$ is the *homotopy fiber* of our bundle over a basepoint, as defined in Example D.3;
- use the orientation isomorphism from Example D.3, as well as the other maps defined in the example and their properties;
- assume a working knowledge of the convergence theorem of the Serre spectral sequence for local coefficients (see Theorem E.9), as well as the notation from Appendix E.1;
- use the product structure of the Serre spectral sequence with local coefficients, as given in Appendix E.1.

We invite the reader to read briefly through these sections and examples in order to become familiar with the notation and conventions used in what follows.

## 4.1 The pushforward map

In this section we will fully address the definition of the pushforward map, using the algebraic tools that we have built up thus far. Using the Serre spectral sequence, we will define a series of maps whose composition is the pushforward map. It is by understanding these maps that we will understand the subtle need for the $d$-odd hypothesis.

We will first map $H^n(E;\mathbb{Z})$ into the $E_\infty$ page of the spectral sequence on the $2d$-row. Then, as we will show, we can inject $E_\infty^{n-2d,2d}$ into $E_2^{n-2d,2d}$. Finally, because

$H^{2d}(W_g) \cong \mathbb{Z}$, we have an isomorphism $E_2^{n-2d,2d} \xrightarrow{\sim} H^{n-2d}(B;\mathbb{Z})$. The composition of these three maps gives us our desired $\pi_! : H^*(E;\mathbb{Z}) \to H^{*-2d}(B;\mathbb{Z})$. The only hypothesis we have is that our bundle be oriented.

**Lemma 4.1** ([Gri13, Lemma 3.3])**.** *The filtration on cohomology is such that, for all $d$, the following two equalities hold:*

$$F^{n-2d}H^n(E;\mathbb{Z}) = H^n(E;\mathbb{Z}) \tag{4}$$

$$F^{n-d}H^n(E;\mathbb{Z}) = F^{n-2d+1}H^n(E;\mathbb{Z}). \tag{5}$$

*Proof.* It suffices to show that $E_2^{n-q,q} = 0$ for $q > 2d$ and for $2d > q > d$, which comes immediately because our fiber $W_g$ only has nonzero cohomology in degrees 0, $d$ and $2d$, giving that both the $E_2$ and $E_\infty$ pages only have nonzero entries on rows 0, $d$ and $2d$. □

This gives us our first map, because by the convergence theorem of the Serre spectral sequence, Lemma 4.1 gives us the following map:

$$H^n(E;\mathbb{Z}) = F^{n-2d}H^n(E;\mathbb{Z}) \overset{p}{\twoheadrightarrow} F^{n-2d}H^n(E;\mathbb{Z})/F^{n-2d+1}H^n(E;\mathbb{Z}) = E_\infty^{n-2d,2d} \tag{6}$$

which allows us to map $H^n(E;\mathbb{Z})$ into $E_\infty^{n-2d,2d}$, as desired.

**Lemma 4.2** ([Gri13, Lemma 3.4])**.** $E_\infty^{n-2d,2d} \subset E_2^{n-2d,2d}$.

*Proof.* Recall that our fiber $W_g$ only has nonzero cohomology in degrees 0, $d$ and $2d$, giving that both the $E_2$ and $E_\infty$ pages only have nonzero entries on rows 0, $d$ and $2d$. Note also that from the $E_2$ page and onwards, all the differentials in the spectral sequence go in the down-right direction. Since our groups in question are on row $2d$, in the decomposition given by Theorem E.9 as $E_\infty^{n-2d,2d} = Z^{n-2d,2d}/B^{n-2d,2d}$ for subgroups $B^{n-2d,2d}$ and $Z^{n-2d,2d}$ of $E_2^{n-2d,2d}$, of necessity $B^{n-2d,2d} = 0$, so $E_\infty^{n-2d,2d} = Z^{n-2d,2d} \subset E_2^{n-2d,2d}$, as desired. □

Lemma 4.1 gives us the obvious inclusion map, as desired. Note that this result depends on the groups $E_\infty^{n-2d,2d}$ and $E_2^{n-2d,2d}$ being on row $2d$, or having only trivial groups in rows above them, which is why $\pi_!$ decreases in degree by the dimension of the fiber.

Finally, we remind the reader of the orientation isomorphism $\mathcal{H}^{2d}(M) \xrightarrow{\sim} \mathbb{Z}$ given before. This induces an isomorphism $E_2^{n-2d,2d} = H^{n-2d}(B;\mathcal{H}^{2d}(M)) \xrightarrow{\sim} H^{n-2d}(B;\mathbb{Z})$, which is the final map in our composition. The definition of $\pi_!$ follows.

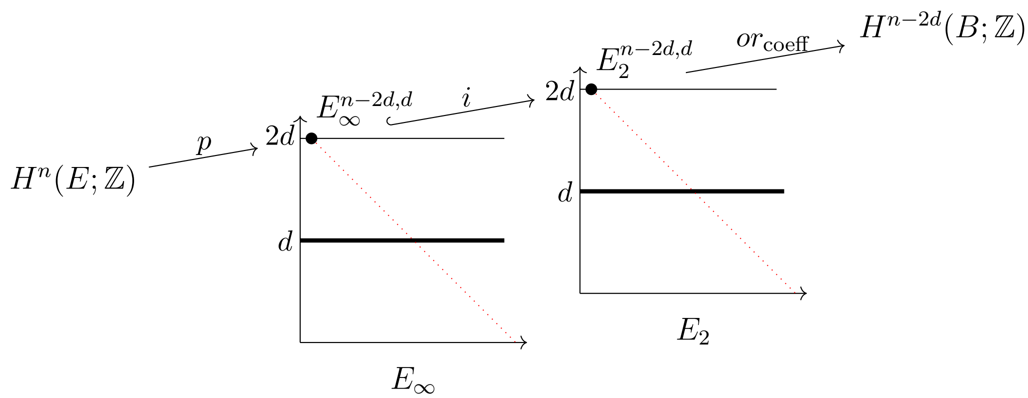


**Figure 1:** The pushforward map through the $E_\infty$ and $E_2$ pages of the Serre spectral sequence. The horizontal bars on the pages represent the $2d$, $d$ and $0$ rows—the only rows which have nontrivial groups. The diagonal lines are there to show the groups whose indexes sum to $d$.

**Definition 4.3.** *The pushforward map on cohomology, $\pi_! : H^*(E;\mathbb{Z}) \to H^{*-2d}(B;\mathbb{Z})$, is the map that makes the following diagram commute:*

$$\begin{tikzcd}
H^n(E;\mathbb{Z}) = F^{n-2d}H^n(E;\mathbb{Z}) \arrow[r, "\pi_!"] \arrow[d, two heads, "p"] & H^{n-2d}(B;\mathbb{Z}) \\
E_\infty^{n-2d,2d} \arrow[r, hook, "i"] & E_2^{n-2d,2d} \arrow[u, "\wr"', "or_{coeff}"]
\end{tikzcd}$$

*where $i$ is the inclusion map given by Lemma 4.2, $p$ is the map from Equation 6, and $or_{coeff}$ is the orientation isomorphism described at the start of this section.*

We have included Figure 1 to illustrate the pushforward map as it goes through the Serre spectral sequence. As we alluded to before, that our definition given is equivalent to the definition given in the beginning using a fiber integral is a result in [Boa70].

We make the statement here that the pushforward is natural with respect to pull-backs, meaning that it takes characteristic classes to characteristic classes. This is vital if we are going to find any useful information about the characteristic classes in $H^*(B;\mathbb{Z})$ via those in $H^*(E;\mathbb{Z})$.

## 4.2 Proof of Theorem 3.1

We are now prepared to give a proof of Theorem 3.1, which we restate here for the convenience of the reader. As we have mentioned before, the results from [Gri13] are true when the fiber in question is $(d-1)$-connected and has the top cohomology isomorphic to $\mathbb{Z}$—i.e. when the fiber's cohomology groups behave similarly to those of $W_g$. Because of the scope of this paper we will focus on the case that our fiber is $W_g$. We invite the interested reader to either make the minor adjustments necessary to prove these results in full generality or to consult [Gri13].

**Theorem 3.1** (Theorem 2.7 in [Gri13]). *Let $d$ be an odd natural number. Let $\pi : E \to B$ be an oriented manifold bundle with fiber $W_g$ and let $a, b \in H^*(E;\mathbb{Z})$ be two classes such that $\pi_!(a) = 0$, $\pi_!(b) = 0$, and $\deg(a)$ is even.*

*Then the classes $\pi_!(a \cup a) \in H^{2\deg(a)-2d}(B;\mathbb{Z})$ and $\pi_!(a \cup b) \in H^{\deg(a)+\deg(b)-2d}(B;\mathbb{Z})$ satisfy the following two relations.*

$$(2g+1)! \cdot \pi_!(a \cup a)^{g+1} = 0 \tag{7}$$

$$(2g+1)! \cdot \pi_!(a \cup b)^{2g+1} = 0 \tag{8}$$

The proof we give relies on the following two main results, which we will prove in the following sections, Sections 4.3 and 4.4. In what follows, $M$ refers to the homotopy fiber of our bundle in question.

**Proposition 4.4** ([Gri13, Proposition 3.8]). *Let $\mathcal{H} = \mathcal{H}^d(M)$. Let $a \in H^{\deg(a)}(E)$ and $b \in H^{\deg(b)}(E)$ be two classes such that $\pi_!(a) = 0$ and $\pi_!(b) = 0$. Then there are $\iota \in H^{\deg(a)-d}(B;\mathcal{H})$ and $\kappa \in H^{\deg(b)-d}(B;\mathcal{H})$ that depend only on $a$ and $b$, respectively, such that $\pi_!(a \cup b)$ is the image of $\iota \otimes \kappa$ under the composition of maps*

$$\begin{array}{ccc} H^{\deg(a)-d}(B;\mathcal{H}) \otimes H^{\deg(b)-d}(B;\mathcal{H}) \xrightarrow{\cup} H^i(B;\mathcal{H}\otimes\mathcal{H}) \xrightarrow{\omega_{coeff}} H^{p+p'}(B;\mathbb{Z}) \\ \iota \otimes \kappa \longmapsto \pi_!(a \cup b) \end{array} \tag{9}$$

*where $i = \deg(a) + \deg(b) - 2d$ and $\omega_{coeff}$ is the map induced by*

$$\omega : \mathcal{H} \otimes \mathcal{H} \xrightarrow{\cup} \mathcal{H}^{2d}(M) \xrightarrow{or} \mathbb{Z}.$$

**Proposition 4.5** ([Gri13, Proposition 4.1]). *Let $\mathcal{H}$ be a twisted coefficient system with fiber $\mathbb{Z}^k$ with $k \leq 2g$. Let $\iota \in H^*(B;\mathcal{H})$ have odd degree. Then*

$$(2g+1)! \cdot \iota^{2g+1} = 0 \in H^{(2g+1)\deg(\iota)}(B;\mathcal{H}^{\otimes 2g+1}).$$

Together, these results give a generalization of the fact that if $a = \iota \cup \kappa$ is an integral cohomology class and at least one of $\iota$ and $\kappa$ has odd degree, then $2a^2 = 0$.

Proposition 4.4 first shows that if $a, b \in H^*(E;\mathbb{Z})$ satisfy the hypotheses of Theorem 3.1, then $\pi_!(a \cup b)$ can be decomposed as the product of two classes in $E_2$ page of the Serre spectral sequence. Note that if $\deg(a)$ is even and $d$ is odd, then our proposition gives us that $\deg(\iota)$ is odd.

Proposition 4.5 shows that, in a similar fashion to the case of integral cohomology, if $\pi_!(a \cup b)$ is the product of two classes on the $E_2$ page, and if at least one of them has odd degree, then there exists some $k$ such that $k \cdot \pi_!(a \cup b)^k = 0$. If we consider everything with rational coefficients, as is the case with our main theorem, the result can be strengthened to say that $\pi_!(a \cup b)$ is nilpotent, and thus zero after we quotient by the nilradical.

We need one final, minor proposition, after which the proof of Theorem 3.1 is quite straightforward.

**Proposition 4.6** (Proposition 4.7 in [Gri13]). *The following diagram commutes.*

$$\begin{tikzcd}
(H^{\deg(a)-d}(B;\mathcal{H}) \otimes H^{\deg(b)-d}(B;\mathcal{H}))^{\otimes l} \arrow[r, "\varphi"] \arrow[d, "\cup"] \arrow[dr, "\cup"] & H^{(\deg(a)-d)\cdot l}(B;\mathcal{H}^{\otimes l}) \otimes H^{(\deg(b)-d)\cdot l}(B;\mathcal{H}^{\otimes l}) \arrow[d, "\cup" ', "\textit{permute coefficients}"] \\
H^i(B;\mathcal{H}\otimes\mathcal{H})^{\otimes l} \arrow[r, "\cup"] \arrow[d, "(\omega_{\mathit{coeff}})^{\otimes l}"] & H^{il}(B;(\mathcal{H}\otimes\mathcal{H})^{\otimes l}) \arrow[d, "(\omega^{\otimes l})_{\mathit{coeff}}"] \\
H^i(B;\mathbb{Z})^{\otimes l} \arrow[r, "\cup"] & H^{il}(B;\mathbb{Z}^{\otimes l} \cong \mathbb{Z})
\end{tikzcd}$$

*where $\varphi$ is the map that first permutes the coordinates of $(\iota\otimes\kappa)\otimes\cdots\otimes(\iota\otimes\kappa)$ and then takes the cup product, as follows:*

$$(\iota\otimes\kappa)\otimes\cdots\otimes(\iota\otimes\kappa) \mapsto (\iota\otimes\cdots\otimes\iota)\otimes(\kappa\otimes\cdots\otimes\kappa) \mapsto (\iota\cup\cdots\cup\iota)\otimes(\kappa\cup\cdots\cup\kappa).$$

*The upper-right, vertical arrow also permutes the coefficients.*

The proof of Proposition 4.6 is a straightforward application of the associativity of the cup product and the fact that the cup product commutes with change of coefficients. To make this more clear, we have a diagram of the images of $(\iota\otimes\kappa)^{\otimes l}$ under these maps:

$$\begin{tikzcd}
(\iota\otimes\kappa)\otimes\cdots\otimes(\iota\otimes\kappa) \arrow[r, mapsto] \arrow[d, mapsto, "\cup"] \arrow[dr, mapsto, "\cup"] & \pm(\iota\cup\cdots\cup\iota)\otimes(\kappa\cup\cdots\cup\kappa) \arrow[d, mapsto, "\cup" ', "\text{permute coeff.}"] \\
(\iota\cup\kappa)\otimes\cdots\otimes(\iota\cup\kappa) \arrow[r, "\cup"] \arrow[d, mapsto, "(\omega_{\text{coeff}})^{\otimes l}"] & \iota\cup\kappa\cup\cdots\cup\iota\cup\kappa \arrow[d, mapsto, "(\omega^{\otimes l})_{\text{coeff}}"] \\
\pi_!(a\cup b)^{\otimes l} \arrow[r, mapsto, "\cup"] & \pi_!(a\cup b)^l
\end{tikzcd}$$

The only ambiguity in the diagram is the upper right-hand corner, where the image of $(\iota \otimes \kappa)^{\otimes l}$ is defined up to a sign, which is $(-1)$ raised to the power of $\sum_{i=1}^{l-1}(\deg(\kappa)\deg(\iota))^i$ by graded commutativity of the cup product.

For our purposes, Proposition 4.6 means that $\pi_!(a \cup b)^{2g+1}$ is the image of $\iota^{2g+1} \cup \kappa^{2g+1}$ under some group homomorphism—specifically, the homomorphism given by the right-hand side of the diagram above. With these results, we proceed with the proof of Theorem 3.1.

*Proof of Theorem 3.1.* Let $a, b \in H^*(E;\mathbb{Z})$ be two classes such that $\pi_!(a) = 0$, $\pi_!(b) = 0$ and $\deg(a)$ is even. By Proposition 4.4 we have the decomposition of $\pi_!(a \cup b)$ as the product of classes $\iota \in H^{\deg(a)-d}(B;\mathcal{H})$ and $\kappa \in H^{\deg(b)-d}(B;\mathcal{H})$. Proposition 4.6 gives us that $\pi_!(a \cup b)^{2g+1}$ is the image of $\iota^{2g+1} \cup \kappa^{2g+1}$ under a group homomorphism (given by the right-hand side of the diagram in Proposition 4.6) which we call $\varphi$.

Then $\iota$ has odd cohomological degree because $\deg(a)$ is even and $d$ is odd. Recall that $H^d(W_g;\mathbb{Z}) = \mathbb{Z}^{2g}$, so $\operatorname{rank}\mathcal{H} = 2g$, and by Proposition 4.5, $(2g+1) \cdot \iota^{2g+1} = 0$. Thus we have

$$(2g+1) \cdot \pi_!(a \cup b)^{2g+1} = \varphi((2g+1) \cdot \iota^{2g+1} \cup \kappa^{2g+1}) = \varphi(0) = 0.$$

Likewise, $\pi_!(a \cup a)^{g+1}$ is the image of $\iota^{g+1} \cup \iota^{g+1} = \iota^{2g+1} \cup \iota$ under a group homomorphism. Since $(2g+1) \cdot \iota^{2g+1} = 0$, we have again that $(2g+1) \cdot \pi_!(a \cup a)^{g+1} = 0$ as desired. □

Through the proofs in the following two sections we will gain our first insight into the necessity of the $d$-odd hypothesis. With that said, we will move straight into the proof of Proposition 4.4.

## 4.3 Proof of Proposition 4.4

The proof of Proposition 4.4 is quite straightforward. We first define a map called the *secondary pushforward*, which maps from the kernel of $\pi_!$. Then the proof of Proposition 4.4 becomes a matter of showing that a diagram commutes. In order to define the secondary pushforward, we only need one small result about $\ker \pi_!$.

**Lemma 4.7** (Lemma 3.6 in [Gri13]). *Let* $(\ker \pi_!)^n := (\ker \pi_!) \cap H^n(E;\mathbb{Z}) \subset H^*(E;\mathbb{Z})$. *If $M$ is highly connected (i.e. is $2d$-dimensional and $(d-1)$-connected), then*

$$(\ker \pi_!)^n = F^{n-d}H^n(E;\mathbb{Z}) \tag{10}$$

*Proof.* Consider the pushforward map given in Definition 4.3, and note that it is the composition of a surjection with two injections. Thus $(\ker \pi_!)^n$ is the kernel of the quotient map

$$H^n(E;\mathbb{Z}) = F^{n-2d}H^n(E;\mathbb{Z}) \twoheadrightarrow F^{n-2d}H^n(E;\mathbb{Z})/F^{n-2d+1}H^n(E;\mathbb{Z}) = E_\infty^{n-2d,2d}$$

and $(\ker \pi_!)^n = F^{n-2d+1}H^n(E;\mathbb{Z})$. The statement of Lemma 4.1 is $F^{n-2d+1}H^n(E;\mathbb{Z}) = F^{n-d}H^n(E;\mathbb{Z})$. $\square$

Now since $(\ker \pi_!)^n = F^{n-d}H^n(E;\mathbb{Z})$, we have a map $(\ker \pi_!)^n \twoheadrightarrow E_\infty^{n-d,d}$ which is precisely the quotient map

$$F^{n-d}H^n(E;\mathbb{Z}) \twoheadrightarrow F^{n-d}H^n(E;\mathbb{Z})/F^{n-d+1}H^n(E;\mathbb{Z})$$

from our spectral sequence. Unfortunately, since we are not working on row $2d$ of the infinity page, we cannot guarantee an injection $E_\infty^{n-d,d} \hookrightarrow E_2^{n-d,d}$. However, by the convergence theorem of the Serre spectral sequence we know that $E_\infty^{n-d,d} = Z^{n-d,d}/B^{n-d,d}$, where both $B^{n-d,d}$ and $Z^{n-d,d}$ are subgroups of $E_2^{n-d,d}$. Thus we have an inclusion

$$E_\infty^{n-d,d} \hookrightarrow \frac{E_2^{n-d,d}}{B^{n-d,d}}.$$

We choose a *function of sets* $\xi : (\ker \pi_!)^n \dashrightarrow E_2^{n-d,d}$ to be a map (choose one) that makes the following diagram commute:

$$\begin{tikzcd}
(\ker \pi_!)^n \arrow[r, equal] \arrow[drrr, dashed, "\xi"] & F^{n-d}H^n(E;\mathbb{Z}) \arrow[r, two heads] & E_\infty^{n-d,d} \arrow[rr, hook] & & \frac{E_2^{n-d,d}}{B^{n-d,d}} \\
 & & & & E_2^{n-d,d} = H^{n-d}(B;\mathcal{H}^d) \arrow[u, two heads]
\end{tikzcd} \tag{11}$$

Note that this is always possible because the vertical arrow in (11) surjects. We use a dashed line for this map to remind the reader that $\xi$ is not a group homomorphism, but a correspondence of sets. We will assume from here on that the choice for $\xi$ is fixed. We will call $\xi$ the *secondary pushforward* as it behaves similarly to the pushforward map, mapping through row $d$ of the Serre spectral sequence instead of row $2d$.

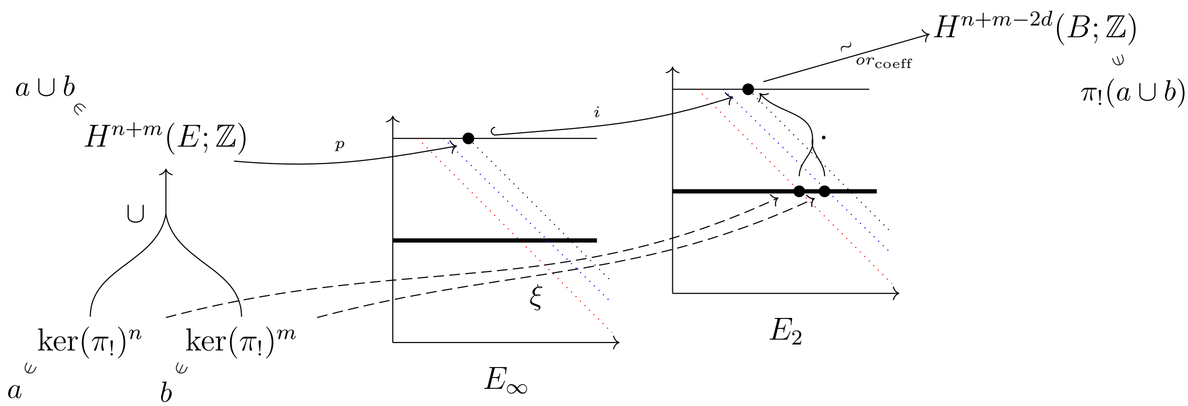


**Figure 2:** We've added a few arrows onto Figure 1 to give a visual image of both $\pi$ and $\xi$ in the context of Proposition 4.8. That the proposition is true is illustrated by the fact that this diagram commutes. The maps along the top, going from $H^n(E;\mathbb{Z})$ to $H^{n-2d}(B;\mathbb{Z})$, compose to be $\pi_!$ as in Figure 1 from Section 4.1. The labels are given in very small letters. The map along the bottom of the diagram is precisely the secondary pushforward, $\xi$, as given in (11). Finally, the vertical arrows are, respectively, the cup product of cohomology on the left and the product on the $E_2$ page on the right.

**Proposition 4.8** (Proposition 3.8 in [Gri13])**.** *Let $a \in (\ker \pi_!)^{p+d}$ and $b \in (\ker \pi_!)^{p'+d}$. Then $\pi_!(a \cup b) \in H^{p+p'}(B;\mathbb{Z})$ is the image of $\xi(a) \otimes \xi(b)$ under the following map:*

$$\begin{array}{ccc} E_2^{p,d} \otimes E_2^{p',d} \xrightarrow{\ \cdot\ } E_2^{p+p',2d} \xrightarrow[or_{coeff}]{\cong} H^{p+p'}(B;\mathbb{Z}) \\ \\ \xi(a) \otimes \xi(b) \longmapsto \pi_!(a \cup b) \end{array} \tag{12}$$

*Proof.* Consider the following diagram:

$$\begin{tikzcd}
(\ker \pi_!)^{p+d} \otimes (\ker \pi_!)^{p'+d} \arrow[rr, "\cup"] \arrow[d, equal] & & H^{p+p'+2d}(E;\mathbb{Z}) \arrow[d, equal] \\
F^pH^{p+d}(E;\mathbb{Z}) \otimes F^{p'}H^{p'+d}(E;\mathbb{Z}) \arrow[d, two heads] & & F^{p+p'}H^{p+p'+2d}(E;\mathbb{Z}) \arrow[d, two heads] \\
E_\infty^{p,d} \otimes E_\infty^{p',d} \arrow[rr, "E_\infty\text{mult.}"] \arrow[d, hook] & & E_\infty^{p+p',2d} \arrow[d, hook] \\
\frac{E_2^{p,d}}{B^{p,d}} \otimes \frac{E_2^{p,p'}}{B^{p',d}} \arrow[rr, red, "E_2\text{mult.}", "(b)"'] & & \frac{E_2^{p+p',2d}}{B^{p+p',2d}} = E_2^{p+p',2d} \arrow[d] \\
E_2^{p,d} \otimes E_2^{p',d} \arrow[u, red, two heads] \arrow[rru, red, "\cdot"'] & & H^{p+p'}(B;\mathbb{Z})
\end{tikzcd}$$

In this diagram, the left column is the composition of maps which we used to define the secondary pushforward $\xi$, and the right column is the composition of maps used to define $\pi_!$. By the convergence theorem of the Serre spectral sequence, the map (b) is well-defined, and the diagram commutes. The equality in the codomain of (b) and $E_2^{p+p',2d}$ comes from Lemma 4.2, which established that $E_\infty^{n-2d,2d} \subset E_2^{n-2d,2d}$. The red triangle from our diagram (included again below) commutes,

$$\begin{tikzcd}
\frac{E_2^{p,d}}{B^{p,d}} \otimes \frac{E_2^{p,p'}}{B^{p',d}} \arrow[r, "E_2\text{mult.}", "\text{(b)}"'] & \frac{E_2^{p+p',2d}}{B^{p+p',2d}} = E_2^{p+p',2d} \\
E_2^{p,d} \otimes E_2^{p',d} \arrow[u, two heads] \arrow[ur] &
\end{tikzcd}$$

which gives us that the map $E_2^{p,d} \otimes E_2^{p',d} \to H^{p,p'}(B;\mathbb{Z})$ from our diagram is precisely the map given in (12), and we have our result. $\square$

Proposition 4.8 is simply a restatement of Proposition 4.4, as the composition of maps in (12), written explicitly in cohomology groups instead of with the $E_2$ page of the spectral sequence, is simply the composition

$$H^p(B;\mathcal{H}) \otimes H^{p'}(B;\mathcal{H}) \xrightarrow{\cup} H^{p+p'}(B;\mathcal{H}\otimes\mathcal{H}) \xrightarrow{\cup_{\text{coeff}}} H^{p+p'}(B;\mathcal{H}^{2d}(M)) \xrightarrow{or_{\text{coeff}}} H^{p+p'}(B;\mathbb{Z}),$$

which is precisely the definition we gave for (9) from Proposition 4.4. In this restatement, $\iota = \xi(a)$ and $\kappa = \xi(b)$.

**Remark:** We wish to pause and note a few insights we can gain from this proof before moving on to the proof of Proposition 4.5. First, the proofs of Proposition 4.4 and all of its supporting lemmas do not rely at all on the degree of $a$ or $b$. Indeed, the degree of $\xi(a)$ is the difference of $\deg(a)$ and $d$. Thinking back to the main theorem which we are looking to prove, we recall that $H^*(BSO(2d);\mathbb{Q})$ is a $\mathbb{Q}$-algebra generated by the Pontryagin and Euler classes. Since each of these has exclusively even degree, the only case in which Proposition 4.4 becomes relevant is when $\deg(a)$ and $\deg(b)$ are both even. As we will see in the next section, this will force $d$ to be odd.

## 4.4 Proof of Proposition 4.5

We will now move on to prove Proposition 4.5 which, together with the proof of Proposition 4.4, will give us valuable insights into our main goal of discovering the necessity of the $d$-odd hypothesis. We will explicitly use the structure of the twisted coefficient system that we have been using in the Serre spectral sequence.

Before giving a proof of the proposition, we need to prove two lemmas. To do this, we remind the reader that the cup product with twisted coefficients is graded commutative in the sense that if $\mathcal{H}$ and $\mathcal{H}'$ are twisted coefficient systems and

$$\tau : \mathcal{H} \otimes \mathcal{H}' \to \mathcal{H}' \otimes \mathcal{H}$$

is the map $a \otimes b \mapsto b \otimes a$, then for $\alpha \in H^p(B; \mathcal{H} \otimes \mathcal{H}')$ and $\beta \in H^q(B; \mathcal{H}' \otimes \mathcal{H})$, we have that

$$\alpha \cup \beta = (-1)^{pq} \tau_{\text{coeff}}(\beta \cup \alpha)$$

(see Example D.3 of Appendix D). This graded commutativity will play essentially the same role in our proof that $\iota^k$ is torsion that the role that graded commutativity plays in the result that $\beta^2$ is torsion if $\beta$ is an integral cohomology class of odd degree.

The proof of Proposition 4.5 will rely on one more notion. Note that for a twisted coefficient system $\mathcal{H}$ and $t \in \mathbb{N}$, $H^{\otimes t}$ is a representation of the symmetric group $S_t$, where the action is given by

$$\sigma \cdot (h_1 \otimes \cdots \otimes h_t) = (h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(t)}).$$

By the same action, $H^*(B; \mathcal{H}^{\otimes t})$ is also an $S_t$-representation. Furthermore, for any $S_t$-representation $V$ we can define the *alternating sub-representation*, $\text{Alt}V$, which consists of vectors $v \in V$ with the property that for all $\sigma \in S_t$,

$$\sigma \cdot v = sgn(\sigma) v.$$

For example, we can consider $\text{Alt}H^*(B; \mathcal{H}^{\otimes t})$.

**Lemma 4.9** (Lemma 4.2 in [Gri13])**.** *If $\iota \in H^{\deg(\iota)}(B; \mathcal{H})$ with $\deg(\iota)$ odd, then $\iota^t \in$ $AltH^*(B; \mathcal{H}^{\otimes t})$.*

*Proof.* Consider the action of a transposition $\tau$ on $\iota^t$, and note that by graded commutativity of the cup product with twisted coefficients, $\tau_{\text{coeff}}(\iota \cup \iota) = (-1)^{\deg(\iota)^2} \iota \cup \iota = -\iota \cup \iota$, and thus

$$\tau_{\text{coeff}}(\iota^t) = -\iota^t \in H^{t \cdot \deg(\iota)}(B; \mathcal{H}^{\otimes t}). \tag{13}$$

Any permutation $\sigma \in S_t$ can be decomposed as a product of transpositions, the number of which is unique modulo 2. If there are $k$ transpositions in a decomposition of $\sigma$, then

$sgn(\sigma) = (-1)^k$. Thus by (13),

$$\sigma_{\text{coeff}}(\iota^t) = (-1)^k \iota^t = sgn(\sigma)\iota^t \in H^{t \cdot \deg(\iota)}(B; \mathcal{H}^{\otimes t})$$

as desired. □

We note here that, just as in the analogous case of integral cohomology, (13) does not hold if $\deg(\iota)$ is not odd. We now prove another lemma, which relates $\mathrm{Alt}H^*(B; \mathcal{H}^{\otimes t})$ and $H^*(B; \mathrm{Alt}\mathcal{H}^{\otimes t})$. After this, the proof of Proposition 4.5 is very straightforward.

**Lemma 4.10** (Lemma 4.3 in [Gri13]). *Suppose that $\alpha \in AltH^{\deg \alpha}(B; \mathcal{H}^{\otimes t})$. Then $t! \cdot \alpha$ is contained in the image of the map $i_{coeff} : H^*(B; Alt\mathcal{H}^{\otimes t}) \to H^*(B; \mathcal{H}^{\otimes t})$.*

*Proof.* Consider first the map on coefficient systems $p : \mathcal{H}^{\otimes t} \to \mathrm{Alt}\mathcal{H}^{\otimes t}$ given by

$$v \stackrel{p}{\mapsto} \sum_{\sigma \in S_t} sgn(\sigma)(\sigma \cdot v).$$

To show that the image of $p$ is genuinely contained in $\mathrm{Alt}\mathcal{H}^{\otimes t}$, it suffices to show that if $\tau$ is a transposition, then

$$\tau \left( \sum_{\sigma \in S_t} sgn(\sigma)(\sigma \cdot v) \right) = sgn(\tau) \left( \sum_{\sigma \in S_t} sgn(\sigma)(\sigma \cdot v) \right).$$

Note that $sgn(\tau\sigma) = sgn(\tau)sgn(\sigma) = -sgn(\sigma)$ and thus that $sgn(\sigma)(\tau \cdot \sigma)v = -sgn(\tau \cdot \sigma)(\tau \cdot \sigma)v = sgn(\tau)sgn(\tau \cdot \sigma)(\tau \cdot \sigma)v$. Thus in the sum,

$$\tau \left( \sum_{\sigma \in S_t} sgn(\sigma)(\sigma \cdot v) \right) = \left( \sum_{\sigma \in S_t} sgn(\sigma)(\tau \cdot \sigma)v \right) = sgn(\tau) \left( \sum_{\sigma \in S_t} sgn(\tau \cdot \sigma)(\tau \cdot \sigma)v \right),$$

as desired.

Now consider $\alpha \in \mathrm{Alt}H^{\deg(\alpha)}(B; \mathcal{H}^{\otimes t}) \subset H^{\deg(\alpha)}(B; \mathcal{H}^{\otimes t})$. By definition, $\sigma_{\text{coeff}} \cdot \alpha = sgn(\sigma)\alpha$, and thus

$$p_{\text{coeff}}(\alpha) = \sum_{\sigma \in S_t} sgn(\sigma)(\sigma_{\text{coeff}} \cdot \alpha) = \sum_{\sigma \in S_t} sgn(\sigma)^2(\alpha) = t! \cdot \alpha.$$

Thus $t! \cdot \alpha = (i \circ p)_{\text{coeff}}(\alpha)$, and we have our desired result. □

*Proof of Proposition 4.5.* Let $\iota \in H^*(B;\mathcal{H})$ have odd degree and suppose that the twisted coefficient system $\mathcal{H}$ has as fiber a free abelian group of rank $< 2g$. Then we have $\mathrm{Alt}\mathcal{H}^{\otimes 2g+1} = 0$. By Lemmas 4.9 and 4.10, we have that $t! \cdot \iota^t$ is in the image of $H^*(B;\mathrm{Alt}\mathcal{H}^{\otimes t})$ under $i_{\mathrm{coeff}}$. So, in particular, $(2g+1)! \cdot \iota^{2g+1} = 0$, as desired. □

## 4.5 Extending Theorem 3.1 to the case that $d$ is even

It is evident from the proof of Lemma 4.9 that Proposition 4.5 relies explicitly on the fact that $\deg(\iota)$ is odd. As we noted above, (13) doesn't hold if $\deg(\iota)$ is even. Going back to Proposition 4.4, if we consider $a \in H^{\deg(a)}(E)$ and $b \in H^{\deg(b)}(E)$ such that $\pi_!(a) = 0$ and $\pi_!(b) = 0$, then in order to guarantee that $(2g+1) \cdot \iota^{2g+1} = 0$ we need either $\deg(a) - d$ or $\deg(b) - d$ to be odd. We don't care about the case that $\deg(a)$ or $\deg(b)$ is odd, because generalized Mumford-Miller-Morita classes have even degree, as they are generated by the $\kappa_{p_i}$ and $\kappa_e$. Thus, $d$ is forced to be odd, and we have our main underlying reason for the necessity of the $d$-odd hypothesis.

We might take this a little further and try to consider a way in which these arguments could be generalized to include the case that $d$ is even. This, however, is unlikely. As we have stated before, these arguments are a generalization of the fact that if $\beta$ is a cohomology class with odd degree, then $2\beta^2 = 0$. It might be true that we can find special properties of the manifold $W_g$ which give us a result analogous to Theorem 3.1 for the case that $d$ is even, but it would not come from the implicit structure of the Serre spectral sequence. Indeed, there is no known explicit structure of the multiplication on the Serre spectral sequence that would allow us to make similar arguments for the $d$-even case.

It may also be true that there exist a class of manifolds other than the $W_g$ for which the structure of the Serre spectral sequence would lend itself to a result such as Theorem 3.1, regardless of the dimension of the manifold. The issue for us is that, because $W_g$ is $(d-1)$-connected and thus only has cohomology in degree 0, $2d$ and $d$, the only way to decompose the image of $\pi_!(a \cup b)$ via the multiplication on the Serre spectral sequence and the images of $a$ and $b$ under some other map is through the $d$-row. Proposition 4.5 makes it so that $d$ is forced to be odd.

Perhaps, given different patterns in cohomology, and thus different nontrivial rows in the cohomology, it would be possible to find maps $\xi'$ such that, under the hypotheses of Theorem 3.1, give us that $\pi_!(a \cup b)$ is the image of $\xi'(a) \otimes \xi'(b)$ under the multiplication map, composed with the orientation map, as we did before.

For example, consider manifold of the form $M^{2d+1} := S^d \times S^{d+1}$. From Example

E.7 in Appendix E we know that $H^k(N;\mathbb{Z}) = \mathbb{Z}$ if $k = 0, d, d+1$ or $2d+1$ and 0 otherwise. In this case, if we could prove the existence of a secondary pushforward $\xi$ such that Proposition 4.4 holds then we Proposition 4.5 would always hold regardless of the parity of $d$ because $d$ and $d+1$ are of opposite parity. This, however, is of no use because $\kappa$ classes are even-dimensional. It seems that, for bundles whose fiber is the product of spheres, the implicit structure of the Serre spectral sequence will yield no fruit for a result such as Theorem 3.1 for the case that $d$ is even. We will likely have to use arguments distinct from those given by [Gri13] in order to prove a result analogous to our main theorem, Theorem 2.1, for the case that $d$ is even.

## 5 Proposition 3.3 revisited

Section 4 was devoted to proving the first (and most significant) result which we used in the proof of Proposition 2.2—which is itself a pillar in the proof of the main theorem, Theorem 2.1. We did this for the purpose of explaining the precise reasons for which Theorem 2.1 requires the $d$-odd hypothesis. This section will be devoted to doing the same for the other supporting result in the proof of Proposition 2.2, Proposition 3.3. We repeat Proposition 3.3 below for the reader's convenience.

**Proposition 3.3** (Proposition 3.3 in [GGRW17]). *Let $d$ be odd, $I = (i_1, ..., i_d)$ be a sequence, $p_I = p_1^{i_1} p_2^{i_2} \cdots p_d^{i_d}$ be the associated monomial in the Pontrjagin classes, and write $|I| = \sum_{j=1}^{d} i_j$. Then:*

*(i) The class $\kappa_{p_I}$ is nilpotent in $R^*(W_g)$.*

*(ii) We have*

$$\chi^{|I|} \cdot \kappa_{ep_I} = \prod_{j=1}^{d} \kappa_{ep_j}^{i_j} \in R^*(W_g)/\sqrt{0} \tag{14}$$

*(iii) If $g \geq 1$ then for all $k > 1$ the classes $\kappa_{e^k}$ is nilpotent in $R^*(W_g)$.*

As the reader might recall, this proposition was used in the proof of Proposition 2.2, which gives the generators of $R^*(W_g)$. We made the comment in Section 3 that the proof of this result relies on the fact that the kappa classes induced by a specific type of characteristic classes are nilpotent in $R^*(W_g)$. These are called the modified Hirzebruch $\mathcal{L}$-classes and denoted $\tilde{\mathcal{L}}_i \in H^{4i}(BSO(2d);\mathbb{Q})$.

In this section we will give an exposition of Proposition 3.3, introducing fully the Hirzebruch $\mathcal{L}$-classes. We will show that the proof of Proposition 3.3 relies on two results. The first of these is Theorem 3.2, as we might expect. Since Theorem 3.2 is proved from Theorem 3.1, this further emphasizes the critical role that Theorem 3.1 and the truth of the statement we quoted earlier in [GGRW17] which said that "we cannot obtain results as conclusive as Theorem 2.1 for $d$ even, as our argument relies on [Gri13]." The second result upon which Proposition 3.3 relies we gave earlier, and we repeat it here.

**Theorem 3.4** (Theorem 2.1 in [GGRW17]). *Let $M$ be a manifold of dimension $2d$. Then the classes $\kappa_{\tilde{\mathcal{L}}_i} \in R^*(M)$ are nilpotent for all natural numbers $i \geq 1$ such that $4i - 2d \neq 0$.*

Before making any comments on the implications that changing the $d$-odd hypothesis to $d$-even on these results, let us first give a definition for the modified Hirzebruch $\mathcal{L}$-classes and give a proof for Proposition 3.3. It is only after we do this that we will be capable of discussing the barriers which prevent us from expanding these results to the case that $d$ is even.

The modified Hirzebursch $\mathcal{L}$-classes $\tilde{\mathcal{L}}_i \in H^{4i}(BSO(2d);\mathbb{Q})$ are a set of characteristic classes which generate the same subring of $H^*(BSO(2d);\mathbb{Q})$ as the Potryagin classes $p_1, p_2, ..., p_d$. In our case, they turn out to be a convenient generating set with which to work.

Consider the graded ring $\mathbb{Q}[x_1, ..., x_d]$ in which all the $x_i$ has degree 2. We define homogeneous symmetric polynomials $\tilde{\mathcal{L}}_i$ by the expression

$$\tilde{\mathcal{L}} = 2^d + \tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_2 + \cdots = \prod_{i=1}^{d} \frac{x_i}{\tanh x_i/2}.$$

Note that, since $\frac{x_i}{\tanh x_i/2}$ is an even function, $\tilde{\mathcal{L}}_i$ is a symmetric polynomial in $x_1^2, x_2^2, ..., x_d^2$. Since the subring of symmetric polynomials is generated by the elementary symmetric polynomials $\sigma_{i,d} := \sigma_{i,d}(x_1^2, ..., x_d^2)$ given by

$$\sigma_{i,d} = \sum_{1 \leq l_1 < \cdots < l_i \leq d} x_{l_1}^2 \cdots x_{l_i}^2,$$

where $\sigma_{0,d} = 1$, the $\tilde{\mathcal{L}}_i$ can be expressed as a polynomial $\tilde{\mathcal{L}}_i(\sigma_{1,d}, ..., \sigma_{d,d})$. Thus we define the *modified Hirzebruch $\mathcal{L}$-classes* to be the following polynomials evaluated at the Pontraygin classes:

$$\tilde{\mathcal{L}}_i := \tilde{\mathcal{L}}_i(p_1, p_2, ..., p_d) \in H^{4i}(BSO(2d);\mathbb{Q}).$$

For the reader familiar with the usual Hirzebursch $L$-classes, $\mathcal{L}$, we note that the modified differ from the usual by $\tilde{\mathcal{L}}_i = 2^{d-2i}\mathcal{L}_i$. As we mentioned before, it is a standard result that the $\tilde{\mathcal{L}}_i$ generate the same subring of $H^*(BSO(2d);\mathbb{Q})$ as the $p_i$ [GGRW17, §2.1].

Parts $(i)$ and $(ii)$ of Proposition 3.3 follow in a relatively straightforward manner from Theorems 3.2 and 3.4.

*Proof of Proposition 3.3* $(i)$ *and* $(ii)$*.* Theorem 3.4 gives us right away that the classes $\kappa_{\tilde{\mathcal{L}}_i}$ are nilpotent for all $i$. Thus by Theorem 3.2, for any $d$-tuple $J = (j_1, ..., j_d)$ and monomial $\tilde{\mathcal{L}}_J = \tilde{\mathcal{L}}_1^{j_1}\tilde{\mathcal{L}}_2^{j_2}\cdots\tilde{\mathcal{L}}_d^{j_d}$, the class $\kappa_{\tilde{\mathcal{L}}_J}$ is nilpotent. Since the $\tilde{\mathcal{L}}_i$ generate the same subring of $H^*(BSO(2d);\mathbb{Q})$ as the $p_i$, any monomial $p_I$ may be written as a polynomial in the $\tilde{\mathcal{L}}_i$ and we have that each $\kappa_{p_I}$ is nilpotent, as desired, and we have $(i)$.

For $(ii)$, let $\pi : E \to B$ be a fiber bundle with fiber $W_g$. For a monomial $p_J$ in the Pontryagin classes, define

$$(p_J)' := \chi e p_J - e\pi^*(\kappa_{ep_J}).$$

Then

$$\kappa_{(p_J)'p_i} = \pi_!(\chi e p_i p_J - e p_i \pi^*(\kappa_{ep_J})) = \chi\kappa_{ep_ip_J} - \kappa_{ep_i}\kappa_{ep_J}$$

because $\pi_!$ is a group homomorphism with respect to addition addition and by the *push-pull formula*:

$$\pi_!(a \cup \pi^*(b)) = \pi_!(a) \cup b$$

(see e.g. [Gri13, Proposition 5.1]). Futhermore, $\kappa_{(p_J)'} = 0$ because $\pi_!(e) = \chi$. That $\pi_!(e) = \chi$ is a well-known fact which follows from the push-pull formula (see e.g [Gri13, Lemma 5.2]).

Then part $(i)$ gives us that $\kappa_{p_i}$ is nilpotent, so by applying Theorem 3.2 we conclude that $\kappa_{(p_J)'p_i}$ is nilpotent. Thus modulo nilpotents, $\chi\kappa_{ep_ip_J} = \kappa_{ep_i}\kappa_{ep_J}$. Since $\pi : E \to B$ was an arbitrary (smooth) fiber bundle with fiber $W_g$, this identity holds in the ring $R^*(W_g)$. We get $(ii)$ by induction. □

We still owe the reader the proof for part $(iii)$. In order to give it, we need to introduce a tautological ring closely related to $R^*(M)$. This is defined by considering objects of the form $(\pi, s)$, where $\pi : E^{k+2d} \to B^k$ is a fiber bundle with fiber $M^{2d}$ and $s : B \to E$ is a section. To fiber bundles with a section we can associate additional characteristic classes $c(\pi, s) := s^*(c(T_\pi))$ for $c$ in a basis $\mathcal{B}$ of $H^*(BSO(2d);\mathbb{Q})$. This gives us a ring homomorphism

$$\mathbb{Q}[c, \kappa_c \,|\, c \in \mathcal{B}] \to H^*(B; \mathbb{Q})$$

which is given by $c \mapsto c(\pi, s)$ and $\kappa_c \mapsto \kappa_c(\pi)$. In a manner analogous to the definition of $R^*(M)$ we let $I_{(M,\star)} \subset \mathbb{Q}[c, \kappa_c \,|\, c \in \mathcal{B}]$ be the ideal of polynomials in the $c$ and $\kappa_c$ which vanish on all smooth fiber bundles equipped with a section. Our desired tautological ring is the quotient ring

$$R^*(M, \star) := \mathbb{Q}[c, \kappa_c \,|\, c \in \mathcal{B}]/I_{(M,\star)}.$$

We only need one more result, taken from [GGRW17], in order to give a proof of $(iii)$. This gives us a convenient connection between $R^*(W_g, \star)$ and $R^*(W_g)$.

**Lemma 5.1** (Lemma 3.2 in [GGRW17])**.** *Let $d$ be odd.*

*(i) For any $c \in \mathcal{B}$, we have*

$$\chi^2 c - \chi\kappa_{ec} - \chi e\kappa_c + \kappa_{e^2}\kappa_c = 0 \in R^*(W_g, \star)/\sqrt{0} \tag{15}$$

*(ii) We have*

$$(\chi - 2)\chi e + \kappa_{e^2} = 0 \in R^*(W_g, \star)/\sqrt{0} \tag{16}$$

*(iii) If $g \neq 1$ then the map*

$$R^*(W_g) \to R^*(W_g, \star)$$

*is injective.*

The map in part $(iii)$ of Lemma 5.1 refers to the natural inclusion map of the $\kappa_c$ classes of $R^*(W_g)$ into $R^*(W_g, \star)$. This map will allow us to deduce that certain elements of $R^*(W_g)$ are nilpotent because their images under the injection are nilpotent. We are now ready to give a proof for part $(iii)$ of Proposition 3.3.

*Proof of Proposition 3.3 part $(iii)$.* First consider the case that $g = 1$, so that $\kappa_e = 0$. Then since

$$\kappa_{e^{2l}} = \kappa_{p_d^l} \text{ and } \kappa_{e^{2l+1}} = \kappa_{ep_d^l},$$

we have from part $(i)$ of this proposition that $\kappa_{p_d^l}$, and thus $\kappa_{e^{2l}}$, is nilpotent. Since $\kappa_e = 0$, by Theorem 3.2 we have that $\kappa_{ep_d^l}$, and thus $\kappa_{e^{2l+1}}$, is nilpotent as desired.

Now let $g > 1$. In this case, $\kappa_e$ is not nilpotent. However, we can write

$$\kappa_{e^{2l}} = \kappa_{p_d^l} \text{ and } \kappa_{e^{2l+1}} = \kappa_{e^3 p_d^{l-1}}$$

and thereby see that the same result holds if we can show that $\kappa_{e^3}$ is nilpotent.

It is now that we will use Lemma 5.1 from above. Recall the standard result that $p_d = e^2$ because, regardless of the parity of $d$, we are working with even-dimensional vector bundles—i.e., we are working with characteristic classes in $H^*(BSO(2d); \mathbb{Q})$. Then using $(i)$ of said lemma with $c = p_d = e^2$,

$$\chi^2 p_d - \chi \kappa_{ep_d} - \chi e \kappa_{p_d} + \kappa_{e^2} \kappa_{p_d} = 0 \in R^*(M_g, \star)/\sqrt{0}.$$

We already know that the class $\kappa_{p_d}$ is nilpotent by part $(i)$ of this proposition, and so we have that $\kappa_{ep_d} = \chi p_d$ modulo nilpotence in $R^*(W_g, \star)/\sqrt{0}$.

We know that, by $(i)$, the class $\kappa_{e^2} = \kappa_{p_d}$ is nilpotent, and so by $(ii)$ of Lemma 5.1 it follows that $e \in R^*(W_g, \star)$ is also nilpotent in the case that $g > 1$. Since $p_d = e^2$ is nilpotent, $\kappa_{ep_d} = \kappa_{e^3}$ is also nilpotent in $R^*(W_g, \star)$. Then since the natural inclusion map $R^*(W_g, \star) \hookrightarrow R^*(W_g)$ is injective (part $(iii)$ of Lemma 5.1), $\kappa_{e^3}$ is nilpotent in $R^*(W_g)$, as desired. □

We invite the reader to carefully consider the proof of Proposition 3.3 and consider where the hypothesis that $d$ be even is used. The reader who does this will note that the only times we used the $d$-odd hypothesis were when we referenced Theorem 3.4, Theorem 3.2 and Lemma 5.1—all of which have the hypothesis that $d$ be odd. We already know that Theorem 3.2 relies on the $d$-odd hypothesis because Theorem 3.1 does. So if we are looking for the root need for the $d$-odd hypothesis, we need to consider Theorem 3.4, Theorem 3.1 and Lemma 5.1.

We will devote Section 6 to considering the obstructions that Theorems 3.1 and 3.4 represent in allowing us to generalize Theorem 2.1. For the remainder of this section, we will turn our attention to Lemma 5.1 to explore the need for the $d$-odd hypothesis. As it turns out, Lemma 5.1 relies on the $d$-odd hypothesis for exactly the same reason that Theorem 3.2 does—because it relies on Theorem 3.1. This gives us the important result that the only two obstructions that we have for generalizing our main theorem, Theorem 2.1, to the case that $d$ is even are found in Theorem 3.1 and Theorem 3.4.

## 5.1 Lemma 5.1 and the $d$-odd hypothesis

We will begin by proving Lemma 5.1, and follow by discussing the need for $d$ to be odd. As it turns out, parts $(i)$ and $(ii)$ of Lemma 5.1 are examples in [Gri13]: part $(i)$ is a slight modification of Example 5.19 in [Gri13], as explained in [GGRW17, Lemma 3.2], and part $(ii)$ is Example 5.17 of [Gri13]. Thus the only proof to give is that of part $(iii)$, in order to show that it does not rely on the $d$-odd hypothesis at all.

*Proof of Lemma 5.1, part (iii).* Let $x \in R^*(W_g)$ such that $x = 0$ in $R^*(W_g, \star)$, and let $\pi : E \to B$ be a fiber bundle with fiber $W_g$. Now consider the pullback of $\pi : E \to B$ over $\pi$, i.e. the pullback $\pi' : \pi^*E \to E$ in the following diagram

$$\begin{tikzcd} \pi^*E \arrow[r] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"'] & B. \end{tikzcd}$$

Note that the fiber bundle $\pi'$ has fiber $W_g$. Using the category theoretic definition of a pullback, we can consider $\pi^*E$ as the equalizer in the following diagram

$$\begin{tikzcd} \pi^*E \arrow[r, hook] & E \times E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ & E \arrow[r, "\pi"] & B. \end{tikzcd}$$

It is easy to see that the diagonal map $\Delta : E \hookrightarrow E \times E$ maps into the image of $\pi^*E$, which gives us a canonical section of $\pi' : \pi^*E \to E$.

Then recall that by hypothesis, $\pi^*(x(\pi)) = 0$, and note that by the *push-pull formula* and the fact that $\pi_!(e(T_\pi)) = \chi = 2 - 2g$, we know that

$$(2 - 2g) \cdot x(\pi) = \pi_!(e(T_\pi)) \cdot \pi^*(x(\pi))).$$

Since $\pi^*(x(\pi)) = 0$, we have that $\pi_!(e(T_\pi)) \cdot \pi^*(x(\pi))) = \pi_!(0) = 0$ and thus that $(2 - 2g) \cdot x(\pi) = 0$. Since $2 - 2g \neq 0$ by hypothesis, we must have that $x(\pi)$ is torsion, but because we have taken coefficients in $\mathbb{Q}$, $x(\pi) = 0$. □

As stated before, the proof of Lemma 5.1, part $(iii)$ does not rely on the hypothesis that $d$ is odd, and so we can disregard it for our purposes. We thus turn our attention to parts $(i)$ and $(ii)$, which come, respectively, from Examples 5.19 and 5.17 in [Gri13].

These examples require some background material which is out of the scope of this paper. We invite the enthusiastic reader to read [Gri13, §5] in order to fully understand the examples. Regardless of the reader's background, however, it is clear by simply reading these two examples from [Gri13] that the reason for the $d$-odd hypothesis in Lemma 5.1 is that these results rely explicitly on Theorem 3.1. Indeed, [Gri13, §5] is devoted to generating relations in the tautological ring by using [Gri13, Theorem 2.7], which is, for us, Theorem 3.1. That these two examples revolve around [Gri13, Theorem 2.7] is stated explicitly in Example 5.17, and in Example 5.19 the author says, "We obtain a relation in the cohomology by applying the second part of Theorem 2.7."

# 6 Moving Forward

The ultimate goal of this paper is to give insights for moving forward with generalizing the main theorem, Theorem 2.1, to the case that $d$ is even. This section will be devoted to drawing on the material from Sections 1 through 5 in order to give a clear, precise picture of the exact reasons which prevent us from using arguments from [Gri13] to give a similar, conclusive result in the case that $d$ is even.

The first thing we ought to do is give an idea for what we might expect, in order to not be chasing results which will end up being false. That is to say, we might naively wish to find a proof of Theorems 3.1 and 3.4 which is independent of parity. Doing so would induce a contradiction because, as we mentioned in Section 2, by Proposition 2.3 we know that if $d$ is even, $R^*(W_g)/\sqrt{0}$ has as a subring $\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_d}]$ when $g > 1$. That is, $\kappa_{ep_d}$ is *not* nilpotent as it is in the case when $d$ is odd. Indeed, by Proposition 2.3 we know that $\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_d}]$ is a subring of $R^*(W_g)/\sqrt{0}$ when $g \neq 1$.

What we need is a set of generators for $R^*(W_g)/\sqrt{0}$ in the case that $d$ is even. If we are to hope for the most natural and simple outcome, we might expect for the following result:

**Conjecture 6.1.** *Let $d$ be even. Then:*

*(i) For any $g$, $R^*(W_g)/\sqrt{0}$ is generated by the $\kappa_{ep_i}$.*

*(ii) $R^*(W_1)/\sqrt{0} = \mathbb{Q}$,*

which carries as a corollary that $R^*(W_g)/\sqrt{0} = \mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_d}]$ if $g \neq 1$. In fact, as we mentioned in Section 2 we know that $R^*(W_1)/\sqrt{0} = R^*(W_1) = \mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_d}]$ regardless of the parity of $d$, and so Conjecture 6.1 holds if $g = 1$.

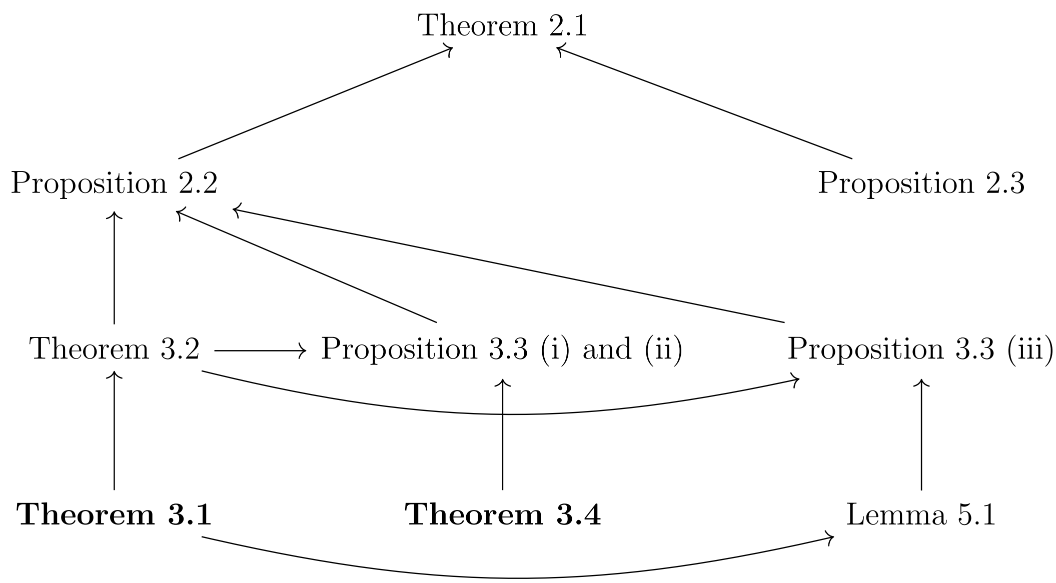


**Figure 3:** The main results mentioned in this document. An arrow between results indicates that the first is used in the proof of the second. The results in bold are the most basic results, supporting the rest, and have the hypothesis that $d$ is odd. For every result besides those two, the need for the $d$-odd hypothesis comes from those results which support it.

Let us recall the proof of Proposition 2.2, which gives the generators for $R^*(W_g)/\sqrt{0}$ when $d$ is odd. It is broken up into three parts:

1. Proposition 3.3 (i) and (ii) imply that for any $g$, $R^*(W_g)/\sqrt{0}$ is generated by the elements $\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_n}$,

2. If $g = 1$ then Theorem 3.2 and Proposition 3.3 (i) imply that each of the $\kappa_{ep_i}$ are nilpotent, giving that $R^*(W_1)/\sqrt{0} = \mathbb{Q}$, and

3. If $g > 1$, Proposition 3.3 (iii) implies that $\kappa_{ep_d}$ is nilpotent.

Right away, we know that Proposition 3.3 (iii) cannot hold for $d$ even, because it implies that $\kappa_{ep_n}$ is nilpotent, which is false. Looking at Figure 3 we can see that if Theorem 3.1 were true for the $d$-even case, then Proposition 3.3 would also be true for $d$-even. And thus we have the following result:

**Corollary 6.2.** *Theorem 3.1 is false if $d$ is even.*

which gives definitive closure to our skepticism that the arguments from [Gri13], specifically those involved with Theorem 3.1, would not generalize to the case that $d$ is even.

More importantly, we note that if Proposition 3.3 (i) and (ii) are true for $d$ even, then we have our desired generators from Conjecture 6.1. However, by again consulting Figure 3 we can see that these results have the $d$-odd hypothesis precisely because their proofs rely on both Theorems 3.1 and 3.4, the first of which is false if $d$ is even. Likewise, the proof that $R^*(W_1)/\sqrt{0} = \mathbb{Q}$ if $d$ is odd relies on Theorem 3.2 and Proposition 3.3 (i) which, respectively, rely on Theorem 3.1 and Theorem 3.4.

# 7 Conclusion

There are a few different directions that one could take research from this point. To prove Conjecture 6.1 or a similar result, the next important step is to find generators of $R^*(W_g)/\sqrt{0}$. We know that we cannot use the results from [Gri13], as they do not generalize to the case that $d$ is odd. We might try to salvage the rest of the structure and try to prove Proposition 3.3 (i) and (ii) for $d$ even, without using Theorem 3.1 and despite the fact that Theorem 3.4 no longer shows that all of the $\kappa_{\tilde{\mathcal{L}}_i}$ are nilpotent. The other option, of course, is to find generators in some other way. Randal-Williams has made some progress by considering torus actions in [RW16] on $S^2 \times S^2$, where he proves that the Krull dimension of $R^*(S^2 \times S^2)$ is 3 or 4. It is likely that any more research on the $d$-even case will require methods and ideas that are out of the scope of this paper.

In order to prove some result for any $d$ we might also try changing coefficients from $\mathbb{Q}$ to $\mathbb{Z}_2$, or considering a slightly different manifold such as $g(S^{d-1} \times S^{d+1})$ for the fiber of our bundles. Both of these ideas could yield techniques and theory which is useful for understanding $R^*(W_g)/\sqrt{0}$ when $d$ is even.

We could also continue with the tools which we have already from the $d$-odd case and change our manifold slightly to be the connected sum of $g$ copies of $S^k \times S^{2n-k}$ for $d$ odd, $g > 1$ and $n \geq k$. From [RW16, Corollary 4.1] we have that the map

$$\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, ..., \kappa_{ep_{n-1}}] \to R^*(g(S^d \times S^{2n-d}))/\sqrt{0}$$

surjects. This is a generalization of the results presented here from [GGRW17], which proved the case that $n = d$. The obvious next step forward is to understand the algebraic relations between these generators $\kappa_{ep_i}$.

At a more basic level, as Grigoriev mentions, the results of [Gri13] are based on the methods of Randal-Williams from [RW12], which are in turn based on Morita's work in [Mor89]. It might be possible that more methods can be found from studying these two papers.

Finally, the results from Ebert and Randal-Williams in [ERW14] suggest that a suitable definition for kappa classes could be made for topological and block bundles. This would allow us to consider the tautological ring in a more general setting. As Grigoriev mentions in [Gri13], this doesn't come immediately and requires some fine-tuning.

# 8 Appendix

Here in the Appendix we will give the necessary background for the main content of this paper. This should enable any reader who has taken a standard introductory course in algebraic topology to be able to read and understand this entire document. For the reader who lacks this basic background, we recommend either [Hat05] or [Mas90] for an excellent introduction to homology and homotopy theory.

# A Fiber Bundles and Fibrations

Fiber bundles and fibrations play a central role in the theory of tautological rings and characteristic classes. They generalize the familiar notion of a covering space in homotopy theory, and also relate to the notion of a sheaf in algebraic geometry. In this section, we will give a brief introduction to fiber bundles and fibrations. We direct the reader who wishes to find a more thorough introduction to the material to read from [Hat09] or [Hus75]. We will begin via a specific (and very important) example of a fiber bundle called a vector bundle.

**Definition A.1** (Vector bundle)**.** *Let $\pi : E \to B$ be a continuous surjection of topological spaces $E$ and $B$. Then $\pi : E \to B$ is a $k$-dimensional real vector bundle if the following conditions are satisfied:*

- *For all $b \in B$, $\pi^{-1}(b)$ is a finite-dimensional real vector space of dimension $k$.*
- *There exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of $B$ such that for all $U_\alpha$ there exist homeomorphisms*
$$\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$
*taking $\pi^{-1}(b)$ to $\{b\} \times \mathbb{R}^n$ via a linear isomorphism.*
- *If $\alpha, \beta \in I$, then the composition $\phi_\beta^{-1} \circ \phi_\alpha : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ is well-defined and satisfies*
$$\phi_\beta^{-1} \circ \phi_\alpha(x, v) = (x, g_{\alpha\beta}(x)v)$$
*for some $GL(k)$-valued function*
$$g_{\alpha\beta} : U \cup V \to GL(k).$$

- *These maps satisfy*

$$g_{\alpha\alpha} = I \quad and \quad g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = I$$

The maps $\phi_\alpha$ are called the *local trivializations* of the vector bundle, the maps $\phi_\beta^{-1} \circ \phi_\alpha$ are called *transition functions*, the last condition is called the *cocycle condition*, and the spaces $E$ and $B$ are called, respectively, the total space and the base space. Thus if a map $\pi : E \to B$ has a vector bundle structure on it, we are saying that locally $\pi$ looks like a projection map of the form $U \times \mathbb{R}^n \to U$. In other words, $E$ is locally the product of $B$ with $\mathbb{R}^n$. We would like to give the reader a few examples with which to understand this definition.

**Example A.2** (Trivial bundle)**.** *The most obvious example of a vector bundle is the natural projection map*

$$B \times \mathbb{R}^n \to B.$$

*The local product structure exists because of the global product structure. This example is called the trivial bundle.*

**Example A.3** (Möbius bundle)**.** *Consider the space $E = ([0,1] \times \mathbb{R})/\sim$, where $\sim$ is the equivalence relation $(0,x) \sim (1,-x)$. There is a retraction $E \to B = S^1$ via the map $(t,x) \mapsto t$, which clearly has the structure of a real vector bundle by taking any open cover of $S^1$ which does not contain the whole space.*

*This is our first nontrivial example of a vector bundle. We can show this by using sections of the vector bundle. A section of a real vector bundle is simply a continuous map $s : B \to E$ such that $\pi \circ s = Id_B$. In other words, $s$ is a section if it maps each point of $B$ into its fiber and does so continuously.*

*Note that the trivial bundle has sections of the form $b \mapsto (b,t)$ for some constant $t$, in particular for which there is no $b \in B$ such that $b \mapsto (b,0)$ if $t \neq 0$. However, in the case of our bundle, it is obvious to see that every section must have a point $b \in S^1$ such that using the isomorphism $\phi : \pi^{-1}(b) \xrightarrow{\sim} \{b\} \times \mathbb{R}$, $\phi \circ s(b) = (b,0)$ because of the intermediate value theorem and the quotient by $(0,x) \sim (1,-x)$. Thus $\pi : E \to B$ is a nontrivial example of a real vector bundle.*

*Note that $E$ is homeomorphic to the Möbius strip with its boundary circle deleted, and so we call this particular real vector bundle the Möbius bundle.*

**Example A.4** (Tautological bundle)**.** *The final example of a real vector bundle which we will give is called the tautological bundle, which is a bundle over a Grassmanian* *manifold $G_n(\mathbb{R}^{n+k})$. Recall that the Grassmanian manifold, as a set, is the set of $n$-dimensional subspaces of $\mathbb{R}^{n+k}$. Thus we can construct a total space $E$ as the set of all $(V, v)$ where $V \in G_n(\mathbb{R}^{n+k})$ is an $n$-dimensional subspace of $\mathbb{R}^{n+k}$ and $v \in V$. We topologize this as a subspace of $G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$ and get a vector bundle structure via the obvious map $\pi : (V, v) \mapsto V$.*

**Example A.5** (Locally free sheaves)**.** *In the context of algebraic geometry and scheme theory, an example of a vector bundle is a locally free sheaf.*

The reader will note that, in all but Example A.3, we did not use any specific properties of $\mathbb{R}$ beyond its vector space structure. There are analagous definitions and examples in the case that we replace $\mathbb{R}$ with $\mathbb{C}$, which is straightforward to work out. We leave it to the reader to work out the examples which correspond to Examples A.2 and A.5 in the complex case.

We can generalize these ideas further to the notion of a *fiber bundle*, in which we replace $\mathbb{R}$ or $\mathbb{C}$ from the examples above with any topological space $X$. Recall that for any map of topological spaces $f : A \to B$, the *fiber* of $f$ over $a \in A$ is simply the preimage $f^{-1}(a)$.

**Definition A.6** (Fiber bundle)**.** *Let $\pi : E \to B$ be a continuous map of topological spaces $E$ and $B$. Then $\pi : E \to B$ is a fiber bundle if the following conditions are satisfied:*

*(i)* *For all $b \in B$, $\pi^{-1}(b)$ is homeomorphic to a fixed topological space $F$*

*(ii)* *There is an open cover $\{U_\alpha\}_{\alpha \in I}$ with isomorphisms*

$$\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$$

*which restricts on fibers to a homeomorphism.*

*(iii)* *If $\alpha, \beta \in I$, then the composition $\phi_\beta^{-1} \circ \phi_\alpha : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ is well-defined and satisfies*

$$\phi_\beta^{-1} \circ \phi_\alpha(x, v) = (x, g_{\alpha\beta}(x)v)$$

*for some $Aut(F)$-valued function*

$$g_{\alpha\beta} : U \cup V \to Aut(F).$$

*(iv) These maps satisfy*

$$g_{\alpha\alpha} = Id \text{ and } g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = Id$$

As before, the $\phi_\beta^{-1} \circ \phi_\alpha$ are called transition functions, the last condition is called the cocycle condition, $E$ is called the total space, $B$ is the base space and we call $F$ the *fiber*. Diagrammatically, a fiber bundle is often drawn as

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d] \\ & B, \end{tikzcd}$$

which gives it the feel of a "short exact sequence of spaces." One can think of fiber bundles intuitively as a quotient, similar to a group quotient, where $F$ takes on the same role that a normal subgroup plays in a group quotient.

As we might hope with any object we define, fiber bundles form a category. Maps of fiber bundles, which we simply call *bundle maps*, are commuting squares

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ B \arrow[r] & B', \end{tikzcd} \tag{17}$$

where $E \to B$ and $E' \to B'$ are fiber bundles and all the maps are continuous. If we fix a base space $B$, we can define a category of fiber bundles over $B$ by defining morphisms to be commuting squares such as (17) with the condition that the map on the bottom row be the identity map.

One important feature of fiber bundles is that they have the homotopy lifting property. Recall that $\pi : E \to B$ has the homotopy lifting property with respect to a space $X$ if, for all homotopies

$$h : [0,1] \times X \to B,$$

if there exists a map $f_0 : \{0\} \times X \to E$ such that $\pi \circ f_0 = h|_{\{0\}\times X}$, then there exists a homotopy $f : [0,1] \times X \to E$ such that $\pi \circ f = h$ and $f|_{\{0\}\times X} = f_0$. A *fibration* is a surjection $\pi : E \to B$ which satisfies the homotopy lifting property with respect to any space.

**Example A.7** (Covering spaces)**.** *If $F$ is discrete we get a fiber bundle which is familiar to any beginning student of algebraic topology, namely a covering space.*

**Example A.8** (Hopf fibration)**.** *One famous example of a fibration which is applicable to homotopy theory is the Hopf fibration*

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^3 \arrow[d, "p"] \\ & S^2. \end{tikzcd}$$

*The map $p : S^3 \to S^2$ can be constructed by giving $S^3$ the structure of the complex subspace of $\mathbb{C}^2$ given by $\{(z_0, z_1) \,|\, |z_0|^2 + |z_1|^2 = 1\}$ and $S^2$ the structure of a subspace of $\mathbb{C} \times \mathbb{R}$ given by $\{(z, x) \,|\, |z|^2 + x^2 = 1\}$. Then $p$ is given by*

$$(z_0, z_1) \mapsto (2z_0 z_1^*, |z_0|^2 - |z_1|^2).$$

*As denoted by the name, it is a standard result that $p : S^3 \to S^2$ is a fibration.*

**Example A.9** (Homotopy Fibration)**.** *In this example we will give an important construction of a fibration, called the homotopy fibration, which we use in Section 4.*

*Given any map $f : E \to B$, we can associate a topological space*

$$E_f := \{(e, p) \,|\, e \in E \text{ and } p : I \to B \text{ such that } p(0) = f(e)\}.$$

*$E_f$ is topologized as a subspace of $E \times B^I$, where $B^I$ is the function space of paths in $B$. There is a natural map, given by*

$$E_f \xrightarrow{f'} B : (e, p) \mapsto p(1),$$

*which we claim is a fibration. To show this, consider a homotopy $g : I \times X \to B$ and take a map $\tilde{g}_0 : X \to E_f$ such that $\tilde{g}_0 \circ f' = g_0$, where $g_0 = g|_{\{0\}\times X}$. We can extend $\tilde{g}_0$ to a homotopy $\tilde{g} : I \times X \to E_f$ which lifts $g$ in the following way: Let $\gamma_x$ be the image of $I \times \{x\}$ under $g$ and we write $(e_x, \sigma_x) = \tilde{g}_0(x)$. We define $\tilde{g}$ to be the map $(t, x) \mapsto (e_x, \tilde{\gamma}_x(t))$, where $\tilde{\gamma}_x(t)$ is the path from $f(e_x)$ to $\gamma_x(t)$ which follows the path $\gamma_x \circ \sigma_x$. It is easy to see that $\tilde{g}$ lifts $g$, and so $f' : E_f \to B$ is a fibration.*

*Note that we can embed $E$ into $E_f$ via the map $e \mapsto (e, p_{const(e)})$, where $p_{const(e)}$ is the constant map $I \to \{e\}$. By contracting the paths in $E_f$, we have that $E_f$ deformation retracts onto $E$ and thus that $E$ and $E_f$ are homotopy equivalent. Furthermore, the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "f"'] & E_f \arrow[dl, "f'"] \\ B. & \end{tikzcd}$$

*We call the fiber of a point $* \in B$ under $f'$ to be the homotopy fiber at $*$. One can think of $E_f$ as a fattening of $E$ which gives us desirable homotopy-theoretic properties, and the homotopy fiber simply as the fiber under this "fattening."*

There are variations of the definition of a fiber bundle, depending on the structure on the spaces which we care about. For example, if we care about a smooth structure on $F$, $E$ and $B$ then we alter the definition slightly to include that all the maps in question be smooth, and that the homeomorphisms be diffeomorphisms. Likewise, bundle maps have the additional condition that they be smooth maps. These are called *smooth bundles*. This leads us to an important example of a vector bundle.

**Example A.10** (Vertical tangent bundle)**.** *Let $M^n$ be a smooth manifold embedded in some $\mathbb{R}^N$, $N > n$. The tangent bundle over $M$ is the subset of $M \times \mathbb{R}^n$ defined by*

$$\{(m, v) \,|\, m \in M \subset \mathbb{R}^N \textit{ and } v \textit{ is in the tangent space of } m\}.$$

*By this definition, every smooth manifold has a unique tangent bundle.*

*Given a smooth map of smooth manifolds, $f : X \to Y$, recall the basic definition from calculus on manifolds that $Df$ is a map from the tangent bundle of $X$ to the tangent bundle of $Y$ which restricts to a linear map on fibers. If $Df$ is a surjection on the tangent space of each point, there exists a natural vector bundle which is associated to $f$, called the vertical tangent bundle, which is defined simply by $T_f := \ker Df$.*

Another variation, which we will introduce in the next section, is called a principal $G$-bundle. We mention one final result which illustrates the importance of fibrations in the study of homotopy groups.

**Theorem A.11** (Theorem 4.41 of [Hat05])**.** *Let $\pi : E \to B$ be a fibration with $B$ path connected. Then there is a long exact sequence of homotopy groups*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

Because the universal cover of $S^1$ is contractible, and a covering map is a fibration, it follows immediately that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_i(S^1) = 0$ for all $i \neq 1$. Then using, for example, the Hopf fibration $S^3 \to S^2$, the long exact sequence of homotopy groups gives that for all $i > 2$, $\pi_i(S^2) \cong \pi_i(S^3)$.

# B Principal $G-$bundles and the Universal Bundle

An important example of a fiber bundle is a principal $G$-bundle. Indeed, it is on the theory of principal $G$-bundles that the theory of characteristic classes, and thus of this entire document, rests. In this section we will give a brief introduction to principal $G$-bundles, following the notes Mitchell and Kottke in, repsectively [Mit01] and [Kot12]. We recommend that the interested reader consult these notes for a more in-depth treatment of the subject.

Let $G$ be a topological group. Then a *left $G$-space* is a topological space $X$ equipped with a continuous left $G$-action $G \times X \to X$. Equivalently, a left $G$-space is a space $X$ equipped with a group homomorphism from $G$ to the group of homeomorphisms $X \to X$. If $X$ and $Y$ are $G$-spaces, then a *$G$-equivariant map* is a map $\phi : X \to Y$ such that $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X$.

Now let $E$ and $B$ be $G$-spaces such that the action of $G$ on $B$ is trivial, and consider a $G$-map $\pi : E \to B$. Then $\pi : E \to B$ is a *principal $G$-bundle* if it satisfies similar local triviality conditions as a fiber bundle. That is, $B$ has an open cover $\{U_\alpha\}_{\alpha\in I}$ such that, for all $\alpha$ there exist $G$-equivariant homeomorpihsms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U \times G$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] \arrow[d, "\pi"] & U \times G \arrow[dl] \\
U. &
\end{tikzcd}$$

Note that the fibers are copies of $G$. The $G$-equivariant homeomorphisms $\phi_\alpha$ could be any map which makes the diagram commute, and so in particular there is not always a canonical identity element in $\pi^{-1}(b)$ for any particular $b$. We call these fibers *$G$-torsors*, which are to groups what affine spaces are to vector spaces.

**Example B.1.** *A normal covering map (i.e. a covering map corresponding to a normal subgroup of the fundamental group of the base space) is a principal $G$-bundle, where $G$ is the group of deck transformations.*

Note that the $G$-equivariant homeomorphisms $\phi_\alpha$ give us a canonical $G$-action on $\pi^{-1}(U_\alpha)$, given by $g \cdot (u, h) = (u, g \cdot h)$. Furthermore, this action is free and transitive. Thus $B$ is the orbit space of the $G$-space $E$, i.e. $B \cong E/G$. We proceed with a basic fact about principal $G$-bundles. For proofs of the results in this section which we do not supply, we direct the reader to [Mit01].

**Lemma B.2.** *Any morphism of principal $G$-bundles is an isomorphism.*

Now let $\pi : P \to B$ be a principal $G$-bundle and consider a map $f : B' \to B$. We allow this to be any continuous map, and then give it the structure of a $G$-equivariant map simply by endowing $B'$ with the structure of a $G$-space via the trivial $G$-action. We can form the category theoretic pullback $P' \equiv f^*P \equiv B' \times_B P$; it is easy to see that $P'$ inherits the structure of a principal $G$-bundle over $B'$ from $P$.

We can immediately note that, as a purely categorical fact, bundle maps $Q \to P'$ are in bijective correspondence with commutative diagrams of the form:

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B. \end{array} \tag{18}$$

By Lemma B.2 we have that $Q$ is isomorphic to $P'$ if and only if there exists a commuting diagram such as (18). Thus, for any given map $B' \to B$ there exists only one possible principal $G$-bundle, up to isomorphism, which will make (18) commute. The following fact, Theorem B.3, allows us to go further and say that for any given homotopy class of maps in $[B', B]$ there exists a principal $G$-bundle unique up to isomorphism which makes (18) commute.

**Theorem B.3.** *Let $P \to B$ be a principal $G$-bundle over an arbitrary space $B$, and suppose that $X$ is a CW-complex. Then if $f, g : X \to B$ are homotopic maps, the pullbacks $f^*P$ and $g^*P$ are isomorphic as principal $G$-bundles over $X$.*

If we are given a principal $G$-bundle $P \to B$ and a CW-complex $B'$, we could in theory classify those principal $G$-bundles which can be achieved as pullbacks of $P \to B$ by the homotopy class of maps which pull it back. A priori, we might run into problems because assigning a principal $G$-bundle the map by which it is a pull-back of $P \to B$ might not even be well-defined. We would like for this notion to be well-defined, and ideally to be able to say that *any* principal $G$-bundle over $B'$ is a pullback of $P \to B$. As it turns out, this can be achieved for certain types of principal $G$-bundles.

**Theorem B.4.** *Let $P \to B$ be a principal $G$-bundle. Then $P$ is weakly contractible if and only if for all CW-complexes $X$, there is a bijective correspondence between $[X, B]$ and principal $G$-bundles over $X$ via the map $f \mapsto f^*P$.*

We can actually relax the condition that $X$ be a CW-complex to just requiring paracompactness, but for our purposes we will simply use CW-complexes. Under the hypotheses of Theorem B.4 hypotheses we call $B$ a *classifying space* of $G$ and $P \to B$ a *universal bundle*. A classifying space of a topological group $G$ is most commonly written as $BG$, while the universal bundle is most commonly written $EG$. As it turns out, classifying spaces and universal bundles of a topological group are unique up to homotopy equivalence, and so $BG$ and $EG$ are often referred to as "the" classifying space (resp. universal bundle) of $G$ when only the homotopy type is needed. In particular, when investigating the homology, cohomology or homotopy groups of a classifying space we can refer to "the" classifying space $BG$. We give another remarkable result, which will allow us, among other things, to define characteristic classes in the next section.

**Theorem B.5.** *Let $G$ be a topological group. There exists a classifying space for $G$.*

We will finish this section by giving the theory of balanced products and structure groups, which will also play an important role in the definition of characteristic classes. Let $W$ be a right $G$-space and $X$ a left $G$-space. Then the *balanced product* $W \times_G X$ is the quotient space $W \times X/\sim$, where $(wg, x) \sim (w, gx)$. (We note here that this is different from a pullback, despite the similarity in notation.) We could equivalently convert $X$ into a right $G$-space by setting $gx = xg^{-1}$ and take the orbit space of $W \times X$ under the diagonal action $(w, x)g = (wg, g^{-1}x)$. Note that if $X = *$ is a point, then $W \times_G *$ is simply the orbit space $W/G$.

Now, suppose that $\pi : E \to B$ is a principal $G$-bundle and let $F$ be a left $G$-space. Since $F \to *$ is $G$-equivariant, and $E \times_G * = B$ we have an induced map $E \times_G F \to E \times_G * = B$ which has the structure of a fiber bundle with fiber $F$. We call a local product of this form a *fiber bundle with fiber $F$ and structure group $G$*. We also call $E \times_G F$ the *associated fiber bundle* to $E$ with fiber space $F$. Because $F$ is a left $G$-space, there is a group homomorphism $G \to \mathrm{Aut}(F)$ corresponding to the left action. In most of our examples we will be interested in the case that $G = \mathrm{Aut}(F)$ and this homomorphism is a group isomorphism.

**Example B.6.** *An $n$-dimensional real vector bundle is a fiber bundle with fiber $\mathbb{R}^n$ and structure group $GL_n(\mathbb{R})$. If we give our vector bundle an inner product, then the structure group will be $O(n)$. If we give our vector bundle an orientation, then the structure group will be $SL_n(\mathbb{R})$ or $SO(n)$. The analogous results hold for complex vector bundles.*

The final result of this section forms the basis for the theory of our next section, characteristic classes.

**Theorem B.7.** *Given any fiber bundle $\pi : E \to B$ with fiber $F$ and structure group $Aut(F)$, there exists a principal $Aut(F)$-bundle $P$ such that $E = P \times_G F$.*

In light of Theorem B.7, consider a fiber bundle $\pi : E \to B$ with fiber $F$. We can choose $\mathrm{Aut}(F)$ depending on the bundle we're interested—for example, if we are looking at a smooth bundle then we let $\mathrm{Aut}(F) = \mathrm{Diff}(F)$, the diffeomorphism group of $F$. We can then find its associated principal $\mathrm{Aut}(F)$-bundle (i.e. the bundle $P$ such that $E = P \times_{\mathrm{Aut}(F)} F$). By Theorem B.4 there is a homotopy class of maps, called the classifying map, which classifies $P$ as a principal $\mathrm{Aut}(F)$-bundle. It is these maps upon which the theory of characteristic classes is built.

## C Characteristic Classes

When studying principal $G$-bundles, a useful tool for studying "how nontrivial" the bundles are (whatever that means) is to study its classifying map. We know from Appendix B that principal $G$-bundles over a CW-complex $X$ are in bijective correspondence with the set $[X, BG]$ of homotopy classes of maps. However, in most cases it is not at all straightforward to find and study these maps. Instead, we use the ever-useful tool of cohomology to study the maps on cohomology which are induced by the maps in $[X, BG]$. This gives us the definition of a characteristic class of a principal $G$-bundle.

**Definition C.1.** *Let $\pi : E \to B$ be a principal $G$-bundle with classifying map $[\varphi] \in [B, BG]$. If $c \in H^*(BG)$, then the characteristic class $c(E) \in H^*(B)$ is the image of $c$ under the map $\varphi^* : H^*(BG) \to H^*(B)$.*

Immediately from this definition, we have that if $E_1$ and $E_2$ are isomorphic principal $G$-bundles over $X$, then their characteristic classes are isomorphic. Likewise, if $E = B \times G$ is the trivial principal $G$-bundle, then its classifying map is nullhomotopic, and so its characteristic classes must be trivial.

Characteristic classes can also be defined for arbitrary fiber bundles with fiber $F$ and structure group $\mathrm{Aut}(F)$. Given such a bundle, by Theorem B.7 we know that there exists a principal $\mathrm{Aut}(F)$-bundle $P$ such that $E = P \times_{\mathrm{Aut}(F)} F$. Since $P$ has a classifying map, we can study the characteristic classes of $P$ via this classifying map. The definition of these characteristic classes is similar to that in Definition C.1.

**Definition C.2.** *Let $\pi : E \to B$ be a fiber bundle with fiber $F$, structure group $Aut(F)$ and associated principal $Aut(F)$-bundle $P$ with classifying map $[\varphi] \in [B, BAut(F)]$. Then if $c \in H^*(BG)$, the characteristic class $c(E) \in H^*(B)$ is defined to be the image of $c$ under the map $\varphi^* : H^*(BAut(F)) \to H^*(B)$.*

There are other formal ways to define characteristic classes, which defines each class as a functor that sends a vector bundle to its characteristic class in $H^*(B)$. These definitions are equivalent to those we have given (see [Kot12]).

We are now going to shift our attention to characteristic classes of vector bundles. As we noted in Appendix B, an $n$-dimensional real vector bundle is a fiber bundle with fiber $\mathbb{R}^n$ and, depending on whether or not we have inner products and orientations, structure group $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$ or $SO(n)$. For the purposes of this paper we will focus our attention on vector bundles with structure group $SO(n)$—i.e. on smooth, oriented vector bundles. A particularly useful example of such vector bundles are tangent bundles of smooth manifolds, which play a vital role in the material of this paper.

For a vector bundle $E \to B$, there are four main types of characteristic classes, which are:

1. Stiefel-Whitney classes $w_i(E) \in H^i(B; \mathbb{Z}/2)$ for a real vector bundle
2. Chern classes $c_i(E) \in H^{2i}(B; \mathbb{Z})$ for a complex vector bundle
3. Pontryagin classes $p_i(E) \in H^{4i}(B; \mathbb{Z})$ for a real vector bundle
4. The Euler class $e(E) \in H^n(B; \mathbb{Z})$ for an oriented $n$-dimensional vector bundle.

As it turns out, we can give a full description of $H^*(BSO(n); \mathbb{Z})$ in terms of the Potryagin and Euler classes. Our focus for the rest of this section will be to give a definition for the Pontryagin and Euler classes and then give the mentioned description. The Pontryagin classes are defined in terms of the Chern classes, which are elements of $H^{2*}(BSO(n))$ and which can in turn be defined by using Schubert cycles. We will not give a description of Chern classes, but direct the interested reader to [Hat09, Chapter 3]. The definition of Pontryagin classes is as follows:

**Definition C.3** (Pontryagin classes)**.** *Let $E \to B$ be a real vector bundle. Then the $k^{th}$ (integral) Pontryagin class of $E$, written $p_k(E)$, is given by*

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(B; \mathbb{Z}),$$

*where $E \otimes \mathbb{C}$ is the complexification of $E$, given by $E \otimes \mathbb{C} = E \oplus iE$.*

The Euler is in some sense a refinement of the Pontryagin classes, as we will see later on. Recall first that the Euler class is for oriented vector bundles. Choosing an orientation for a vector bundle is the same as choosing a generator of $H^n(F, F \setminus F_0; \mathbb{Z})$ for each fiber, where $F_0$ is the zero element. Then via the Thom isomorphism (see, e.g. [Hat09, §3.2]) this gives us a so-called orientation class $u \in H^n(E, E \setminus E_0; \mathbb{Z})$, where $E_0$ is the zero section of $E$. Then using the zero section we get an inclusion $B \hookrightarrow E$, and thus inclusions of pairs

$$(B, \emptyset) \hookrightarrow (E, \emptyset) \hookrightarrow (E, E_0).$$

Then under the induced maps,

$$H^n(E, E \setminus E_0; \mathbb{Z}) \to H^n(E; \mathbb{Z}) \to H^n(B; \mathbb{Z})$$

the image of $u \in H^n(E, E \setminus E_0; \mathbb{Z})$ is the Euler class. It is a standard result that If $E \to B$ is an oriented vector bundle of dimension $2n$ then $e(E)^2 = p_n(E)$.

As we said before, the interests of this paper center on smooth, oriented vector bundles (tangent bundles of smooth manifolds) and their characteristic classes. In particular we are interested in the rational cohomology ring $H^*(BSO(n); \mathbb{Q})$, as $SO(n)$ is the structure group of such vector bundles. In a very convenient fashion, these rings have a very simple presentation.

**Theorem C.4.** *$H^*(BSO(n); \mathbb{Q})$ is the $\mathbb{Q}$-polynomial ring generated by the Potryagin classes $p_i$ and the Euler class $e$. If $n = 2d$, then*

$$H^*(BSO(2d); \mathbb{Q}) = \mathbb{Q}[p_1, ..., p_{d-1}, e].$$

Note that the generators of $H^*(BSO(2d); \mathbb{Q})$ do not include $p_d$ because $e^2 = p_d$.

Characteristic classes of vector bundles are particularly well-understood and we now have excellent tools with which to work with them. This is not true for most other kinds of characteristic classes. In an arbitrary fiber bundle with fiber $F$ and structure group $\mathrm{Aut}(F)$, the cohomology ring $H^*(B\mathrm{Aut}(F); G)$ can be very difficult to understand. In Section 1.1 we define *generalized Miller-Morita-Mumford classes* of a smooth fiber bundle $E \to B$, which are the simplest characteristic classes of smooth fiber bundles to understand. The reason they are so simple is because they draw on the characteristic classes of a natural, associated vector bundle over $E$ and then push them down to classes in $H^*(B)$. That they are characteristic classes of the bundle $E \to B$ means that they are in the image of $H^*(B\mathrm{Diff}(F))$, where $F$ is the fiber, and that they are natural with respect to bundle maps. Other than that, for a smooth manifold $F$, a priori the characteristic classes of a bundle with fiber $F$, given by $H^*(B\mathrm{Diff}(F))$, are not easy to understand. For a more thorough treatment of this subject, we direct the reader to [MS74] or [Kot12].

## D Cohomology with local coefficients

There are two equivalent approaches to defining cohomology with local coefficients, each of which has its strengths. The first is more algebraic, and deals with $\mathbb{Z}\pi_1$-modules, and the second is more topological, where we consider fiber bundles whose fibers are abelian groups. We will need elements of both to properly understand the content of this paper, and so we will give a brief introduction to each one. We follow [KD01, Chapter 5] and [Hat04], and invite the reader who wishes to know more to read from these texts.

We will first proceed with the algebraic definition. We remind the reader that, given a (not necessarily abelian) group $\pi$, the group ring $\mathbb{Z}\pi$ is the ring consisting of linear combinations of elements of $\pi$ with coefficients in $\mathbb{Z}$. Addition is given component-wise:

$$\left(\sum n_i g_i\right) + \left(\sum m_i g_i\right) = \sum (n_i + m_i) g_i.$$

Multiplication is given by the distributive law, using multiplication in $\pi$:

$$\left(\sum_i n_i g_i\right)\left(\sum_j m_j g_j\right) = \sum_{i,j} (n_i m_j)(g_i h_j).$$

Now let $A$ be an abelian group and consider a representation $\rho : \pi \to \mathrm{Aut}_{\mathbb{Z}}(A)$ of $A$. This gives $A$ the structure of a lef $\mathbb{Z}\pi$-module—indeed, left $\mathbb{Z}\pi$-modules are in bijective correspondence with representations $\rho : \pi \to \mathrm{Aut}_{\mathbb{Z}}(A)$ of $A$.

Let $X$ be a path connected and locally path connected topological space which admits a universal cover, and consider $\pi = \pi_1(X)$. Consider the universal cover $\tilde{X}$ of $X$, and note that, via deck transformations, the singular chain complex $S_*(\tilde{X})$ of the universal cover is a right $\mathbb{Z}\pi$-module, where the action of $g \in \mathbb{Z}\pi$ on some $\sigma \in S_*(\tilde{X})$ is given by composing $\sigma$ with the deck transformation $g : \tilde{X} \to \tilde{X}$.

We will now give the definition for cohomology with local coefficients in $A$.

**Definition D.1.** *Given a left $\mathbb{Z}\pi$-module $A$, form the cochain complex*

$$S^*(X; A) = Hom_{\mathbb{Z}\pi}(S_*(\tilde{X}), A).$$

*The cohomology of this complex is called the cohomology of $X$ with local coefficients in $A$ and is written*

$$H^*(X; A).$$

We make the rather intuitive note here that maps and tensor products of local coefficient systems correspond to maps and tensor products of $\mathbb{Z}\pi$-modules.

If we wish to emphasize the representation $\rho : \pi \to \mathrm{Aut}(A)$ corresponding to $A$, we write $H^*(X; A_\rho)$ and call this the *cohomology of $X$ twisted by $\rho$*. We make the fascinating note here that the ordinary cohomology of $X$ with integral coefficients corresponds to the trivial representation (see e.g. [KD01, §5.2]). On the other extreme, if $A$ is a finitely generated free $\mathbb{Z}\pi$-module, then the cohomology of $X$ twisted by $\rho$ is the cohomology of $\tilde{X}$ with (untwisted) coefficients in $\mathbb{Z}$. In this case, $A$ corresponds to the *tautological representation* $\rho : \pi \to \mathrm{Aut}(\mathbb{Z}\pi)$, given by

$$\rho(g) = \left(\sum m_h h \mapsto \sum m_h gh\right).$$

As it turns out, the (untwisted) cohomology of any cover of $X$ can be obtained by the correct choice of local coefficients. From the algebraic point of view, this fact is quite remarkable. It becomes more intuitive in light of the more topological approach to local coefficients.

We now wish to give the definition of local coefficients via this approach, using a local coefficient system. Recall that a *local coefficient system* over $X$ is a fiber bundle over $X$ whose fiber is a discrete abelian group $A$ with structure group $G \leq \mathrm{Aut}(A)$. (Note that this implies that a local coefficient system is a covering map.)

Let $p : E \to X$ be a system of local coefficients, and denote $p^{-1}(x)$ by $E_x$. We will construct a cochain complex by first defining a chain complex with differential $\partial$, and then using $\partial$ to define a cochain complex with differential $\delta$. The chain complex $S_k(X; E)$ is defined in the obvious way, by taking formal sums of the form

$$\sum_{i=1}^{m} a_i \sigma_i$$

where $\sigma_i : \Delta^k \to X$ is a singular $k$-simplex, and $a_i$ is an element of $E_{\sigma_i(e_0)}$ where $e_0 = (1, 0, ..., 0)$. That is, $S_k(X; E)$ is the abelian group of formal sums of singular $k$-simplices $\sigma$ which have coefficients in the fiber of the baspoint $\sigma(e_0)$. We can consider

$S_k(X;E)$ as a subgroup of the direct sum

$$\bigoplus_{x\in X} S_k(X;E_x).$$

The differential $\partial$ is a bit tricky to define because we have to take into account the fact that every $k$-simplex has one face that does not contain $e_0$. This means that one of the face maps, which are given by

$$f_m^k(t_0,t_1,\cdots,t_{k-1})=(t_0,\cdots,t_{m-1},0,t_m,\cdots,t_{k-1}),$$

does not preserve the basepoint—specifically, $f_0^k$ does not preserve the basepoint, as $(1,0,\cdots,0)\mapsto(0,1,0,\cdots,0)$.

To remedy this, we will induce an isomorphism of groups over the fibers of $e_0\in\Delta^k$ and the image of $e_0$ under $f_0^k$ via the path $\gamma_\sigma$ given by $\sigma(t,1-t,0,0,\cdots,0)$. Then $\gamma_\sigma$ defines an isomorphism of groups $E_{\sigma(0,1,\cdots,0)}\xrightarrow{\sim}E_{\sigma(1,0,0,\cdots,0)}$ because $p:E\to X$ has a discrete fiber, and is thus a covering map. Then we define our differential $\partial: S_k(X;E)\to S_{k-1}(X;E)$ by

$$a\sigma\mapsto\gamma_\sigma(a)(\sigma\circ f_0^k)+\sum_{m=1}^{k}(-1)^m a(\sigma\circ f_m^k).$$

We did not give it, but there is an algebraic definition of homology with local coefficients (which is a natural analogue to the definition we gave for cohomology) to which this is equivalent. Now that we have $S_k(X;E)$ and $\partial$, we are ready to give our topological definition of cohomology with local coefficients.

We let $S^k(X;E)$ be the set of maps $c$ such that

$$(\sigma:\Delta^k\to X)\mapsto c(\sigma)\in E_{\sigma(e_0)}.$$

Note that $S^k(X;E)$ is an abelian group. We define the boundary operator $\delta: S^k(X,E)\to S^{k+1}(X;E)$ as follows:

$$(\delta c)(\sigma)=(-1)^k\left(\gamma_\sigma^{-1}(c(\partial_0\sigma))+\sum_{i=1}^{k+1}(-1)^i c(\partial_i\sigma)\right).$$

It is not difficult to verify that $\delta$ and, for that matter, $\partial$ are differentials. The remarkable result is that these two definitions are equivalent. For a proof of this, see [KD01, Chapter 5]. The theorem is the following:

**Theorem D.2** ([KD01, Theorem 5.9]). *The cohomology of the chain complex $(S^*(X;E),\delta)$ equals the cohomology $H^*(X;A_\rho)$, where $\rho:\pi_1 X\to Aut(A)$ is the homomorphism determined by the local coefficient system $p:E\to X$.*

Both of these perspectives on cohomology with local coefficients will lend perspective to this paper. We note here the rather intuitive fact that if $\mathcal{A}$ and $\mathcal{B}$ are systems of local coefficients then a map $f:\mathcal{A}\to\mathcal{B}$ induces *covariantly* a map $H^*(X,\mathcal{A})\to H^*(X,\mathcal{B})$. The map of cohomology induced by a map $f:\mathcal{A}\to\mathcal{B}$ of twisted coefficient systems is written $f_{\mathrm{coeff}}$. We will finish this section with an important example, which plays a vital role in Section 4.

**Example D.3.** *Let $\pi:E\to B$ be a fibration, and let $M$ be the homotopy fiber at a basepoint $*\in B$. The homotopy fiber has the form*

$$M=\{(e,p)\,|\,e\in E \text{ and } p \text{ is a path from } \pi(e) \text{ to } *\}$$

*and so $\pi_1(B,*)$ has a natural left action given by $\gamma\cdot(e,p)=(e,\gamma\circ p)$. This induces a left action of $\pi_1(B,*)$ on the cohomology groups $H^i(M;\mathbb{Z})$ for all $i$, and so $H^i(M;\mathbb{Z})$ is a (left) $\mathbb{Z}[\pi_1(B,*)]$-module. We will write the system of twisted coefficients corresponding to $H^i(M;\mathbb{Z})$ as $\mathcal{H}^i(M)$.*

*If the fibers of $\pi:E\to B$ are $d$-dimensional manifolds, an orientation is a choice of an isomorphism $or:\mathcal{H}^{2d}\xrightarrow{\sim}\mathbb{Z}$, where $\mathbb{Z}$ is the untwisted coefficient system which corresponds to the trivial action of $\pi_1(B,*)$ on $\mathbb{Z}$ (i.e. the $\mathbb{Z}[\pi_1(B,*)]$-module given by $(\sum\sigma_i n_i)\cdot n=(\sum n_i)\cdot n$).*

*Finally, recall that maps and tensor products of twisted coefficient systems correspond to maps and tensors of $\mathbb{Z}[\pi_1(B,*)]$-modules, and so in particular the cup product on cohomology induces a map*

$$\cup:\mathcal{H}^i\otimes\mathcal{H}^j\to\mathcal{H}^{i+j}. \tag{19}$$

*This map is distinct from the cup product on cohomology with twisted coefficients, which is the most natural map*

$$\cup:H^i(B;\mathcal{A})\otimes H^j(B;\mathcal{B})\to H^{i+j}(B;\mathcal{A}\otimes\mathcal{B}), \tag{20}$$

*which tensors the coefficients as would be intuitively expected. Since any map $\mathcal{A}\to\mathcal{B}$ of twisted coefficient systems induces a map $H^*(B;\mathcal{A})\to H^*(B;\mathcal{B})$, by composing (20)* *and the map induced by (19), we can get a map*

$$H^i(B;\mathcal{H}^k)\otimes H^j(B;\mathcal{H}^l)\to H^{i+j}(B;\mathcal{H}^k\otimes\mathcal{H}^l)\to H^{i+j}(B;\mathcal{H}^{k+l}). \tag{21}$$

*The map in (20) has the following properties:*

- *It is associative in the sense that the order of composition is unimportant in the map involving three systems of local coefficients such as*

$$H^*(X;\mathcal{A})\otimes H^*(X;\mathcal{B})\otimes H^*(X;\mathcal{C})\to H^*(X;\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C})$$

- *it is natural with respect to changing coefficients in the sense that, given maps of coefficient systems $f:\mathcal{A}\to\mathcal{C}$ and $g:\mathcal{B}\to\mathcal{D}$ then the following diagram commutes*

$$\begin{tikzcd} H^*(X;\mathcal{A})\otimes H^*(X;\mathcal{C}) \arrow[r, "f_{coeff}\otimes g_{coeff}"] \arrow[d, "\cup"'] & H^*(X;\mathcal{B})\otimes H^*(X;\mathcal{D}) \arrow[d, "\cup"] \\ H^*(X;\mathcal{A}\otimes\mathcal{C}) \arrow[r, "(f\otimes g)_{coeff}"] & H^*(X;\mathcal{B}\otimes\mathcal{D}) \end{tikzcd}$$

- *it is graded commutative in the sense that if $\mathcal{H}$ and $\mathcal{H}'$ are twisted coefficient systems and*

$$\tau:\mathcal{H}\otimes\mathcal{H}'\to\mathcal{H}'\otimes\mathcal{H}$$

  *is the map $a\otimes b\mapsto b\otimes a$, then for $\alpha\in H^p(B;\mathcal{H}\otimes\mathcal{H}')$ and $\beta\in H^q(B;\mathcal{H}'\otimes\mathcal{H})$, we have that*

$$\alpha\cup\beta=(-1)^{pq}\tau_{coeff}(\beta\cup\alpha)$$

# E The Serre Spectral Sequence

In this section we will introduce the notion of a spectral sequence, as derived from an exact couple, and then give as a specific example the Serre spectral sequence. We will introduce the derivation for the spectral sequence of a filtered topological space, following [Hat04]. For a more in-depth treatment of spectral sequences and their derivation, we direct the reader to [Gal16].

We will begin with the notion of an exact couple and then, following [Hat04], will give an example.

**Definition E.1.** *An exact couple consists of abelian groups $A$ and $B$, and maps $i:A\to A$, $j:A\to E$ and $k:E\to A$ such that the following triangle is exact at each of* *its three corners*

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

We can define a map $d = j \circ k : A \to A$ and thereby define a *derived couple.*

**Definition E.2.** *The derived couple of the exact couple given above consists of abelian groups $A' = i(A) \subset (A)$ and $B'$ the homology of $E$ with respect to $d$, and maps $i' = i|_{A'}$, $j' : A' \to E'$ defined by $j'(i(a)) = [ja]$ and $k' : E' \to A'$ given by $k'[e] = k(e)$ which form the following commutative triangle*

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

.

Checking that the maps $j'$ and $k'$ are well-defined is a simple exercise of diagram chasing, as is proving the following lemma.

**Lemma E.3.** *The derived couple of an exact couple is exact.*

This gives us a sequence of abelian groups $E, E', E'', ...$ which may or may not stabilize. This sequence, along with the differentials $d, d', d'', ...$ is called a *spectral sequence*. This is generally formulated as a sequences of pages $E^r$ with differentials $d_r : E^r \to E^r$ such that $d_r^2 = 0$. In this sense the spectral sequences is a more complicated (and correspondingly more powerful) analogue to the long exact sequence. Just like long exact sequences are used to express relationships between (co)homology groups of different spaces, spectral sequences are powerful tools for relating the cohomology groups of more complicated structures for which a long exact sequence is insufficient.

For example, the Adams spectral sequence is used for computing stable homotopy groups, the Leray spectral sequence for sheaf cohomology, and the Grothendieck spectral sequence is useful for computing the composition of derived functors. The Serre spectral sequence, which we are most interested in, is useful for expressing the relationship between the (co)homology groups of spaces in a fiber bundle.

In a spectral sequence, the groups $E^r$ are typically expressed as a direct sum of countably many groups which are indexed by $\mathbb{Z}^2$. Because of that, $E^r$ is usually drawn over $\mathbb{R}^2$ with a group at each lattice-point of $\mathbb{Z}^2$. We call $E^r$ the $r^{th}$ *page* of the spectral sequence, where the $(p, q)^{th}$ direct summand of $E^r$ is written $E^r_{p,q}$.

Differentials are also expressed in terms of the summands. Specifically,

$$d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

so one can tell the page by the direction of the differentials.

Typically, for a spectral sequence to be useful, for each $(p, q)$ there should be some $n \in \mathbb{N}$ such that $E^m_{p,q} = E^n_{p,q}$ if $m \geq n$. That is, $E^k_{p,q}$ should stabilize after large enough $k$. The page with only stable entries is called the *infinity page*, and is written $E^\infty_{p,q}$. In some cases, the infinity page corresponds to an actual page $E^r$. In others, the infinity page is the limit page as $r$ goes to infinity (hence the name). The relationship on which the spectral sequence sheds light is typically expressed by giving the formula for $E^k_{p,q}$ for some $k$ and then giving the $(p, q)^{th}$ diagonal on the infinity page. For example, the Serre spectral sequence for homology is written

$$E^2_{p,q} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E)$$

while the sequence for cohomology is written

$$E^{p,q}_2 = H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)$$

(note the change of the subscripts and superscripts), where $E \to B$ is a fiber bundle with fiber $F$. What is meant by the notation is that there exists a spectral sequence such that the $2^{nd}$ page is given by the formula on the left-hand side, and such that

$$\bigoplus_{a+b=p+q} E^\infty_{a,b} = H^{p+q}(E).$$

Just as in the case of results about long exact sequences, despite the amazing power of spectral sequences the differentials are, in general, unknown and very difficult to understand. In the instances which we use the Serre spectral sequence, the pages have enough trivial entries (i.e. enough $(p, q)$ such that, for example, $E^2_{p,q} = 0$) that the calculations become quite straightforward and virtually no information about the nature of the differentials is necessary.

In the remainder of this section we will construct the exact couple which gives rise to the Serre spectral sequence, and then give the main theorem which shows its properties. We will then give some examples to illustrate its power and diversity. There is a version of the Serre spectral sequence for both homology and cohomology; since we are working only with cohomology in this document we will only introduce the theorem with respect to cohomology. The version for homology is very similar; the interested reader can read, for example, in [Hat04].

In order to construct the exact couple, we must first introduce the idea of a *filtered topological space.*

**Definition E.4.** *A filtration of a topological space $X$ is a collection of subspaces $\{X_\alpha \subset X\}_{\alpha \in I}$ such that $I$ is a totally ordered set, and if $\alpha < \beta$ then $X_\alpha \subset X_\beta$. If $X$ has a filtration, then $X$ is a filtered topological space.*

Examples of filtered topological spaces are CW-complexes, where the filtration is given by the skeleta, as well as simplicial copmlexes. Also, given a continuous map $f : X \to \mathbb{R}$ there exists a natural filtration $X_\alpha = \{f^{-1}(\beta) \mid \beta \leq \alpha\}$.

Now consider a fiber bundle $X \to B$ with fiber $F$ and $B$ a CW-complex. Then $B$ has a natural filtration given by the skeleta of $B$. We write $B_i$ for the $i$-skeleton of $B$, and we can define a filtration for $X$, given by $X_i = \pi^{-1}(B_i)$. For each $i$, we can take the long exact sequence associated to the pair $(X_{i+1}, X_i)$, which is

$$\cdots \to H^n(X_{i+1}, X_i) \to H^n(X_{i+1}) \to H^n(X_i) \to H^{n+1}(X_{i+1}, X_i) \to \cdots .$$

By carefully arranging this long exact sequence for each pair $(X_i, X_{i-1})$, we can fit them together neatly in a *staircase diagram*

$$\begin{tikzcd}
H^{n-1}(X_i) \arrow[r] \arrow[d] & H^n(X_{i+1}, X_i) \arrow[r] & H^n(X_{i+1}) \arrow[r] \arrow[d] & H^{n+1}(X_{i+2}, X_{i+1}) \arrow[r] & H^{n+1}(X_{i+2}) \arrow[d] \\
H^{n-1}(X_{i-1}) \arrow[r] \arrow[d] & H^n(X_i, X_{i-1}) \arrow[r] & H^n(X_i) \arrow[r] \arrow[d] & H^{n+1}(X_{i+1}, X_i) \arrow[r] & H^{n+1}(X_{i+1}) \arrow[d] \\
H^{n-1}(X_{i-2}) \arrow[r] & H^n(X_{i-1}, X_{i-2}) \arrow[r] & H^n(X_{i-1}) \arrow[r] & H^{n+1}(X_i, X_{i-1}) \arrow[r] & H^{n+1}(X_i)
\end{tikzcd}$$

where the red indicates the long exact sequence for the pair $(X_{i+1}, X_i)$. A staircase diagram as given above determines an exact couple by letting $A$ be the direct sum of all the absolute cohomology groups $H^n(X_i)$ and letting $E$ be the the direct sum of all the relative cohomology groups $H^n(X_{i+1}, X_i)$. The maps $i$, $j$ and $k$ which form the exact couple are the maps forming the long exact sequences in the staircase diagram.

The rather remarkable result is that the spectral sequence derived from this exact couple relates the cohomology of $X, B$ and $F$ in the following way:

**Theorem E.5** (Convergence theorem of the Serre spectral sequence for cohomology)**.** *Let $X \to B$ be a fibration with fiber $F$ such that $B$ is path connected, and let $G$ be an abelian group. If $\pi_1(B)$ acts trivially on $H_*(F;G)$, then there is a spectral sequence $\{E_r^{p,q}, d_r\}$, as defined above, such that:*

*(a)* $E_2^{p,q} = H^p(B; H_q(F;G))$

*(b)* $\bigoplus_{p+q=n} E_\infty^{p,q} \cong H^n(X;\mathbb{Z})$

*(c)* $d_r : E_r^{p,q} \to E_r^{p-(r+1),q+r}$ *where* $E_{r+1}^{p,q}$ *is the homology of* $E_r^{p,q}$ *with respect to* $d_r$*.*

*(d) stable terms* $E_\infty^{p,n-p}$ *are isomorphic to the successive quotients* $F_p^n/F_{p-1}^n$ *with respect to a filtration of* $H^*(X)$

The filtration referred to in (d) is given by

$$H^*(X;\mathbb{Z}) = \cdots = F^0H^*(X;\mathbb{Z}) \supset F^1H^*(X;\mathbb{Z}) \supset \cdots$$

where $F^iH^*(X;\mathbb{Z}) := \ker(H^*(X;\mathbb{Z}) \to H^*(X_{i-1};\mathbb{Z}))$. We would like to give a few examples which will give the reader an idea of how the spectral sequence might be used.

**Example E.6.** *We show that if $Y$ is weakly contractible, then $H^*(X \times Y; G) \cong H^*(X;G)$. This is not difficult to verify via the Künneth theorem, but we include the example nonetheless to give the reader some intuition on how the spectral sequence operates.*

*Consider the trivial bundle $X \times Y \to X$ with fiber $Y$. Using the formula from Theorem E.5, we have $E_2^{p,q} = H^p(X; H^q(Y;G))$. Since $Y$ is contractible, the $E_2$-page of the spectral sequence only has nontrivial groups if $q = 0$. This gives us a row of nontrivial groups, namely $E_2^{p,0} = H^p(X;G)$.*

*The differentials from the $E_2$-page onward are not horizontal, and so all the differentials are trivial and we have that the $E_2$-page is the $E_\infty$-page and thus $H^p(X \times Y;G) \cong H^p(X;G) = H^p(X;H^0(Y;G))$, as desired.*

**Example E.7** (Products of spheres)**.** *In this example we will compute $H^*(S^d \times S^{d+1};\mathbb{Z})$ for all $d$. Note that $S^d \times S^{d+1}$ fits into the trivial fiber bundle*

$$\begin{tikzcd} S^d \arrow[r, hook] & S^d \times S^{d+1} \arrow[d] \\ & S^{d+1}. \end{tikzcd}$$

*This gives us an $E_2$ page with only four nontrivial entries, being $E_2^{0,0}$, $E_2^{0,d}$, $E_2^{d+1,0}$ and $E_2^{d+1,d}$, all of which are $\mathbb{Z}$. In this case, the $E_2$ page is the $E_\infty$ page because nowhere do the differentials map between any two of these four nontrivial entries. Thus $H^k(S^d \times S^{d+1}) = \mathbb{Z}$ if $k = 0, d, d+1, 2d+1$ and $0$ otherwise.*

There is an additional structure on the Serre spectral sequence for cohomology which is extremely important. This is that it is *multiplicative*, in the sense that there are bilinear maps

$$\cup : E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$$

induced by the cup product on cohomology. The differential follows the Leibniz rule with regards to this multiplication. This is very useful, as we will see here in Example E.8.

**Example E.8.** *This example is considerably more interesting than the previous two: we will show the cup product structure of $\mathbb{C}P^n$, which cannot be done using exact sequences. Recall that $H^i(\mathbb{C}P^n) = \mathbb{Z}$ if $i$ is even, $i \leq 2n$, and $0$ otherwise. We first start with the fibration*

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^{2n+1} \arrow[d] \\ & \mathbb{C}P^n \end{tikzcd}$$

*which gives us, by the formula from Theorem E.5, the only nontrivial rows are 0 and 1. The first two rows of the first quadrant have the form*

$$\begin{array}{ccccccc} \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} \end{array}$$

*where the only differentials which could be nonzero are of the form $d_2 : E_2^{p,1} \to E_2^{p+2,0}$, mapping $\mathbb{Z} \to \mathbb{Z}$. Since we know the cohomology of $S^{2n+1}$ is $\mathbb{Z}$ if $i = 0$ or $2n+1$, and $0$ otherwise, each of these maps must be isomorphisms for $0 \leq p \leq 2n-2$ because taking the homology of the $E_2$-page with respect to $d_2$ must leave nontrivial groups only in $E_3^{0,0}$ and $E_3^{2n,1}$. This gives us the following image of the $E_2$-page.*

$$\begin{tikzcd} \mathbb{Z} \arrow[drr, "\cong"] & 0 & \mathbb{Z} \arrow[drr, "\cong"] & 0 & \cdots & \mathbb{Z} \arrow[drr, "\cong"] & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & \mathbb{Z} & 0 & \mathbb{Z} \end{tikzcd}$$

*We will use the cup product structure on the spectral sequence to deduce the cup product structure on $\mathbb{C}P^n$, whose cohomology groups lie on the bottom row of the diagram.*

*First, take* $1 \in E_2^{0,0} = H^0(\mathbb{C}P^n; \mathbb{Z})$ *and choose a generator* $a \in E_2^{0,1} = H^0(\mathbb{C}P^n; \mathbb{Z})$*. Let* $x \in E_2^{2,0} = H^2(\mathbb{C}P^n; \mathbb{Z})$ *be the image of* $a$ *under* $d_2$*. Now consider* $xa \in E_2^{2,1} = H^2(\mathbb{C}P^n; \mathbb{Z})$ *and note that by the Leibniz rule*

$$d_2(xa) = (d_2 x) \cdot a + x \cdot (d_2 a).$$

*Then* $d_2 x = d_2(d_2 a) = 0$ *and* $d_2 a = x$ *so*

$$d_2(xa) = (d_2 x) \cdot a + x \cdot (d_2 a) = x^2,$$

*which generates* $E_2^{4,0} = H^4(\mathbb{C}p^n; \mathbb{Z})$*. Likewise,*

$$d_2(x^2 a) = (d_2 x^2) a + x^2 d_2(a) = x^3,$$

*and so on. By continuing this process we deduce that*

$$H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[x]/(x^{2n+1}).$$

## E.1 The Serre spectral sequence with twisted coefficients

We actually do not use the Serre spectral sequence as stated in Theorem E.5 in this paper. Instead, we use a slightly more general version of the theorem using local (or twisted) coefficients as defined in Appendix D. We will present a version of the convergence theorem of the Serre spectral sequence using the local coefficient systems $\mathcal{H}^i$ defined in Example D.3 from Appendix D. We remind the reader that we consider a bundle $\pi : E \to B$ with fiber $M$.

**Theorem E.9** (Convergence theorem with local coefficients, [McC00, Theorem 5.2])**.** *There exists a spectral sequence such that:*

*(a)* $E_2^{p,q} = H^p(B; \mathcal{H}^q(M))$*, and*

*(b)* $E_\infty^{p,q} = F^p H^{p+q}(E; \mathbb{Z}) / F^{p+1} H^{p+q}(E; \mathbb{Z})$

*where the* $F^i H^*(E)$ *are defined as above.*

We note that, as stated in [Gri13, Theorem 3.2], the description of $E_\infty^{p,q}$ is equivalent to that of repeatedly taking subquotients using the differentials of the spectral sequence. Since repeatedly taking subquotients results in a subquotient of the original group, there are subgroups $B^{p,q} \subset Z^{p,q} \subset E_2^{p,q}$ such that

$$E_\infty^{p,q} = Z^{p,q}/B^{p,q}.$$

Here we are using the notation in [Gri13] which, as noted there, is a slight abuse of notation as $Z^{p,q} =$ ker(differentials out of the $(p,q)$ terms) and $B^{p,q}$ =image(differentials into $(p,q)$ terms).

There is an additional property which is very convenient with regards to the filtration, which is that the filtration of $H^*(E;\mathbb{Z})$ given by

$$H^*(E;\mathbb{Z}) = F^0H^*(E;\mathbb{Z}) \supset F^1H^*(E;\mathbb{Z}) \supset F^2H^*(E;\mathbb{Z}) \supset \cdots$$

respects the cup product. That is, the cup product restricts to a map

$$F^pH^*(E;\mathbb{Z}) \otimes F^qH^*(E;\mathbb{Z}) \to F^{p+q}H^*(E;\mathbb{Z}).$$

We will finish here by making some comments on the cup product structure on the Serre spectral sequence. The first is that the filtration of $H^*(E;\mathbb{Z})$ given by

$$H^*(E;\mathbb{Z}) = F^0H^*(E;\mathbb{Z}) \supset F^1H^*(E;\mathbb{Z}) \supset F^2H^*(E;\mathbb{Z}) \supset \cdots$$

respects the cup product, i.e. the cup product restricts to a map

$$F^pH^*(E;\mathbb{Z}) \otimes F^qH^*(E;\mathbb{Z}) \to F^{p+q}H^*(E;\mathbb{Z}).$$

The second is that the product on the $E_2$ page of the spectral sequence, with $E_2^{p,q} := H^p(B;\mathcal{H}^q(M))$, is the following composition of maps:

$$H^p(B;\mathcal{H}^q(M)) \otimes H^{p'}(B;\mathcal{H}^{q'}(M)) \to H^{p+p'}(B;\mathcal{H}^p(M) \otimes \mathcal{H}^{q'}(M)) \to H^{p+p'}(B;\mathcal{H}^{q+q'}(M)).$$

We described these in (21) from Appendix D, i.e. the first map is the cup product of cohomology with twisted coefficients, given in (20), and the second is the map induced by (19). These facts are used in the proof of Theorem 3.1.

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