An Introduction to Local Coefficients

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There are two equivalent approaches to defining cohomology with local coefficients, each of which has its strengths. The first is more algebraic, and deals with $\mathbb{Z}\pi_1$-modules, and the second is more topological, where we consider fiber bundles whose fibers are abelian groups. We will need elements of both to properly understand the content of this paper, and so we will give a brief introduction to each one. We follow [KD01, Chapter 5] and [Hat04], and invite the reader who wishes to know more to read from these texts.

We will first proceed with the algebraic definition. We remind the reader that, given a (not necessarily abelian) group π, the group ring $\mathbb{Z}\pi$ is the ring consisting of linear combinations of elements of π with coefficients in $\mathbb{Z}$. Addition is given component-wise:

$$\left(\sum n_i g_i\right) + \left(\sum m_i g_i\right) = \sum (n_i + m_i) g_i.$$

Multiplication is given by the distributive law, using multiplication in π:

$$\left(\sum_i n_i g_i\right)\left(\sum_j m_j g_j\right) = \sum_{i,j} (n_i m_j)(g_i h_j).$$

Now let A be an abelian group and consider a representation $\rho : \pi \to \mathrm{Aut}_{\mathbb{Z}}(A)$ of A. This gives A the structure of a lef $\mathbb{Z}\pi$-module—indeed, left $\mathbb{Z}\pi$-modules are in bijective correspondence with representations $\rho : \pi \to \mathrm{Aut}_{\mathbb{Z}}(A)$ of A.

Let X be a path connected and locally path connected topological space which admits a universal cover, and consider $\pi = \pi_1(X)$. Consider the universal cover $\tilde{X}$ of X, and note that, via deck transformations, the singular chain complex $S_*(\tilde{X})$ of the universal cover is a right $\mathbb{Z}\pi$-module, where the action of $g \in \mathbb{Z}\pi$ on some $\sigma \in S_*(\tilde{X})$ is given by composing σ with the deck transformation $g : \tilde{X} \to \tilde{X}$.

We will now give the definition for cohomology with local coefficients in A.

Definition 0.1. *Given a left $\mathbb{Z}\pi$-module A, form the cochain complex*

$$S^*(X;A) = Hom_{\mathbb{Z}\pi}(S_*(\tilde{X}), A).$$

The cohomology of this complex is called the cohomology of X with local coefficients in A and is written

$$H^*(X;A).$$

We make the rather intuitive note here that maps and tensor products of local coefficient systems correspond to maps and tensor products of $\mathbb{Z}\pi$-modules.

If we wish to emphasize the representation $\rho : \pi \to \mathrm{Aut}(A)$ corresponding to A, we write $H^*(X; A_\rho)$ and call this the *cohomology of X twisted by ρ*. We make the fascinating note here that the ordinary cohomology of X with integral coefficients corresponds to the trivial representation (see e.g. [KD01, §5.2]). On the other extreme, if A is a finitely generated free $\mathbb{Z}\pi$-module, then the cohomology of X twisted by ρ is the cohomology of $\tilde{X}$ with (untwisted) coefficients in $\mathbb{Z}$. In this case, A corresponds to the *tautological representation* $\rho : \pi \to \mathrm{Aut}(\mathbb{Z}\pi)$, given by

$$\rho(g) = \left(\sum m_h h \mapsto \sum m_h gh\right).$$

As it turns out, the (untwisted) cohomology of any cover of X can be obtained by the correct choice of local coefficients. From the algebraic point of view, this fact is quite remarkable. It becomes more intuitive in light of the more topological approach to local coefficients.

We now wish to give the definition of local coefficients via this approach, using a local coefficient system. Recall that a *local coefficient system* over X is a fiber bundle over X whose fiber is a discrete abelian group A with structure group $G \leq \mathrm{Aut}(A)$. (Note that this implies that a local coefficient system is a covering map.)

Let $p : E \to X$ be a system of local coefficients, and denote $p^{-1}(x)$ by E_x. We will construct a cochain complex by first defining a chain complex with differential ∂, and then using ∂ to define a cochain complex with differential δ. The chain complex $S_k(X;E)$ is defined in the obvious way, by taking formal sums of the form

$$\sum_{i=1}^{m} a_i \sigma_i$$

where $\sigma_i : \Delta^k \to X$ is a singular k-simplex, and a_i is an element of $E_{\sigma_i(e_0)}$ where $e_0 = (1, 0, ..., 0)$. That is, $S_k(X;E)$ is the abelian group of formal sums of singular

k-simplices σ which have coefficients in the fiber of the baspoint $\sigma(e_0)$. We can consider $S_k(X;E)$ as a subgroup of the direct sum

$$\bigoplus_{x\in X} S_k(X;E_x).$$

The differential ∂ is a bit tricky to define because we have to take into account the fact that every k-simplex has one face that does not contain e_0. This means that one of the face maps, which are given by

$$f_m^k(t_0, t_1, \cdots, t_{k-1}) = (t_0, \cdots, t_{m-1}, 0, t_m, \cdots, t_{k-1}),$$

does not preserve the basepoint—specifically, f_0^k does not preserve the basepoint, as $(1, 0, \cdots, 0) \mapsto (0, 1, 0, \cdots, 0)$.

To remedy this, we will induce an isomorphism of groups over the fibers of $e_0 \in \Delta^k$ and the image of e_0 under f_0^k via the path γ_σ given by $\sigma(t, 1-t, 0, 0, \cdots, 0)$. Then γ_σ defines an isomorphism of groups $E_{\sigma(0,1,\cdots,0)} \xrightarrow{\sim} E_{\sigma(1,0,0,\cdots,0)}$ because $p : E \to X$ has a discrete fiber, and is thus a covering map. Then we define our differential $\partial : S_k(X;E) \to S_{k-1}(X;E)$ by

$$a\sigma \mapsto \gamma_\sigma(a)(\sigma \circ f_0^k) + \sum_{m=1}^{k} (-1)^m a(\sigma \circ f_m^k).$$

We did not give it, but there is an algebraic definition of homology with local coefficients (which is a natural analogue to the definition we gave for cohomology) to which this is equivalent. Now that we have $S_k(X;E)$ and ∂, we are ready to give our topological definition of cohomology with local coefficients.

We let $S^k(X;E)$ be the set of maps c such that

$$(\sigma : \Delta^k \to X) \mapsto c(\sigma) \in E_{\sigma(e_0)}.$$

Note that $S^k(X;E)$ is an abelian group. We define the boundary operator $\delta : S^k(X,E) \to S^{k+1}(X;E)$ as follows:

$$(\delta c)(\sigma) = (-1)^k \left(\gamma_\sigma^{-1}(c(\partial_0 \sigma)) + \sum_{i=1}^{k+1} (-1)^i c(\partial_i \sigma) \right).$$

It is not difficult to verify that δ and, for that matter, ∂ are differentials. The remarkable result is that these two definitions are equivalent. For a proof of this, see

[KD01, Chapter 5]. The theorem is the following:

Theorem 0.2 ([KD01, Theorem 5.9])**.** *The cohomology of the chain complex* $(S^*(X;E),\delta)$ *equals the cohomology* $H^*(X;A_\rho)$*, where* $\rho:\pi_1 X\to Aut(A)$ *is the homomorphism determined by the local coefficient system* $p:E\to X$*.*

Both of these perspectives on cohomology with local coefficients will lend perspective to this paper. We note here the rather intuitive fact that if $\mathcal{A}$ and $\mathcal{B}$ are systems of local coefficients then a map $f:\mathcal{A}\to\mathcal{B}$ induces *covariantly* a map $H^*(X,\mathcal{A})\to H^*(X,\mathcal{B})$. The map of cohomology induced by a map $f:\mathcal{A}\to\mathcal{B}$ of twisted coefficient systems is written f_{coeff}. We will finish this section with an important example.

Example 0.3. *Let* $\pi:E\to B$ *be a fibration, and let* M *be the homotopy fiber at a basepoint* $*\in B$*. The homotopy fiber has the form*

$$M=\{(e,p)\,|\,e\in E \textit{ and } p \textit{ is a path from } \pi(e) \textit{ to } *\}$$

and so $\pi_1(B,*)$ *has a natural left action given by* $\gamma\cdot(e,p)=(e,\gamma\circ p)$*. This induces a left action of* $\pi_1(B,*)$ *on the cohomology groups* $H^i(M;\mathbb{Z})$ *for all* i*, and so* $H^i(M;\mathbb{Z})$ *is a (left)* $\mathbb{Z}[\pi_1(B,*)]$*-module. We will write the system of twisted coefficients corresponding to* $H^i(M;\mathbb{Z})$ *as* $\mathcal{H}^i(M)$*.*

If the fibers of $\pi:E\to B$ *are* d*-dimensional manifolds, an orientation is a choice of an isomorphism* $or:\mathcal{H}^{2d}\xrightarrow{\sim}\mathbb{Z}$*, where* $\mathbb{Z}$ *is the untwisted coefficient system which corresponds to the trivial action of* $\pi_1(B,*)$ *on* $\mathbb{Z}$ *(i.e. the* $\mathbb{Z}[\pi_1(B,*)]$*-module given by* $(\sum\sigma_i n_i)\cdot n=(\sum n_i)\cdot n$*).*

Finally, recall that maps and tensor products of twisted coefficient systems correspond to maps and tensors of $\mathbb{Z}[\pi_1(B,*)]$*-modules, and so in particular the cup product on cohomology induces a map*

$$\cup:\mathcal{H}^i\otimes\mathcal{H}^j\to\mathcal{H}^{i+j}. \tag{1}$$

This map is distinct from the cup product on cohomology with twisted coefficients, which is the most natural map

$$\cup:H^i(B;\mathcal{A})\otimes H^j(B;\mathcal{B})\to H^{i+j}(B;\mathcal{A}\otimes\mathcal{B}), \tag{2}$$

which tensors the coefficients as would be intuitively expected. Since any map $\mathcal{A}\to\mathcal{B}$ *of twisted coefficient systems induces a map* $H^*(B;\mathcal{A})\to H^*(B;\mathcal{B})$*, by composing (2)* *and the map induced by (1), we can get a map*

$$H^i(B;\mathcal{H}^k)\otimes H^j(B;\mathcal{H}^l)\to H^{i+j}(B;\mathcal{H}^k\otimes\mathcal{H}^l)\to H^{i+j}(B;\mathcal{H}^{k+l}). \tag{3}$$

The map in (2) has the following properties:

- *It is associative in the sense that the order of composition is unimportant in the map involving three systems of local coefficients such as*

$$H^*(X;\mathcal{A})\otimes H^*(X;\mathcal{B})\otimes H^*(X;\mathcal{C})\to H^*(X;\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C})$$

- *it is natural with respect to changing coefficients in the sense that, given maps of coefficient systems $f:\mathcal{A}\to\mathcal{C}$ and $g:\mathcal{B}\to\mathcal{D}$ then the following diagram commutes*

$$\begin{array}{ccc} H^*(X;\mathcal{A})\otimes H^*(X;\mathcal{C}) & \xrightarrow{f_{coeff}\otimes g_{coeff}} & H^*(X;\mathcal{B})\otimes H^*(X;\mathcal{D}) \\ \cup\downarrow & & \downarrow\cup \\ H^*(X;\mathcal{A}\otimes\mathcal{C}) & \xrightarrow{(f\otimes g)_{coeff}} & H^*(X;\mathcal{B}\otimes\mathcal{D}) \end{array}$$

- *it is graded commutative in the sense that if $\mathcal{H}$ and $\mathcal{H}'$ are twisted coefficient systems and*

$$\tau:\mathcal{H}\otimes\mathcal{H}'\to\mathcal{H}'\otimes\mathcal{H}$$

  is the map $a\otimes b\mapsto b\otimes a$, then for $\alpha\in H^p(B;\mathcal{H}\otimes\mathcal{H}')$ and $\beta\in H^q(B;\mathcal{H}'\otimes\mathcal{H})$, we have that

$$\alpha\cup\beta=(-1)^{pq}\tau_{coeff}(\beta\cup\alpha)$$

References

[Hat04] Allen Hatcher. Spectral sequences in algebraic topology. 1 2004.

[KD01] Paul Kirk and James F. Davis. *Lecture Notes in Algebraic Topology (Graduate Studies in Mathematics, 35)*. American Mathematical Society, 2001.