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Introduction to Algebriac $K-$Theory

Introduction

In this document, we will define a sequence of abelian groups, written $K_0(R)$, $K_1(R)$ and $K_2(R)$, which are called the $K-$groups of a ring R with unit, and which are algebraic invariants of R. While there are well-defined higher K-groups, and even negative $K-$groups, we will focus only on these three.

In Sections 1 and 2, we will give thoroughly ring-theoretic definitions of $K_0(R)$ and $K_1(R)$ for any ring with unit. However, as its name suggests, algebriac $K-$theory has many connections to topological $K-$theory, an extraordinary cohomology theory of compact Hausdorff topological spaces. Our main goal in Section 3 will be to generalize the notion of $K_0(R)$ and $K_1(R)$ from rings to arbitrary categories with exact sequences. Then in Section 4 we will expose the link between algebraic and topological $K-$theory by looking at $K_0(R)$ of the ring of continuous $\mathbb{R}-$ or $\mathbb{C}-$valued maps on a topological space.

We will end our analysis of algebraic $K-$theory in Section 5 by giving a definition of $K_2(R)$ for an arbitrary ring with unit, and showing it to be both functorial in R and abelian.

This exposition closely follows the book *Algebraic K-Theory and its Applications*, by Jonathan Rosenberg [2], except for Section 4 which draws somewhat from Allen Hatcher's (yet unfinished) book *Vector Bundles and K-Theory* [1].

1 The Functor K_0

Let R be a ring. Unless otherwise specified, throughout this document we will use the convention that R refers to a ring with unit, and that I refers to a ring without unit. All ring homomorphisms will be unit-preserving, the word "module" will refer to a left-module, and all modules will be assumed to be finitely generated.

We assume the reader is familiar with elementary homological algebra, in particular projective and injective modules and resolutions, and homology of chain complexes.

Then let R be a ring, and let $Proj(R)$ be the set of finitely generated, projective $R-$modules up to isomorphisms. Recall that a finitely generated $R-$module is projective iff it is a direct summand of R^n for some $n \in \mathbb{N}$. Also, if P_1 and P_2 are projective $R-$modules such that $P_1 \oplus Q_1 \cong R^n$ and $P_2 \oplus Q_2 \cong R^m$ then $P_1 \oplus P_2$ is a projective $R-$module, because by canonical isomorphism,

$$(P_1 \oplus P_2) \oplus (Q_1 \oplus Q_2) \cong (P_1 \oplus Q_1) \oplus (P_2 \oplus Q_2) \cong R^{m+n}.$$

Furthermore, $0 \oplus R \cong R$, so 0 is projective, the $\oplus$ operation is (canonically) associative, and $\oplus$ is a commutative operation because by canonical isomorphism, $P_1 \oplus P_2 \cong P_2 \oplus P_1$. Thus, aside from being a set, $Proj(R)$ is a commutative semigroup under the operation $\oplus$ and with the identity element the zero module.

Proposition 1.1. *Let S be a commutative semigroup with unit 0. Then there exists a unique abelian group $G(S)$ and a canonical map $S \hookrightarrow G(S)$ which is universal with respect to semigroup homomorphisms $\phi : S \longrightarrow H$, where H is a group.*

Proof. Define $G(S)$ to be the free abelian group on elements of S, modulo the relation that, for $a, b, c \in S$, $a + b = c$ in $G(S)$ iff a + b = c in S. Our canonical map is the obvious inclusion $S \hookrightarrow G(S)$. Then for any group H, a semigroup homomorphism $\phi : S \longrightarrow H$ can be uniquely lifted to a group homomorphism $\phi' : G(S) \longrightarrow H$ by mapping $s^{-1} \mapsto \phi(s)^{-1}$ for all $s \in S$.

Uniqueness comes immediately because $G(S)$ is universal. □

The group $G(S)$ is often called the **Grothendieck group** or the **group completion** of S. [2]

There is an alternate, slightly more concrete construction of $G(S)$, which is similar to the construction of the field of fractions of an integral domain. In this case, we construct an abelian group G by taking formal differences of elements of S with the addition operation $[r_1 - s_1] + [r_2 - s_2] = [(r_1 + r_2) - (s_1 + s_2)]$, and saying that $r_1 - s_1 \sim r_2 - s_2$ if, for some $t \in S$, $r_1 + s_2 + t = s_1 + r_2 + t$. In this construction, the identity element is the class $[r - r]$, for any $r \in S$, because $s - t \sim (s + r) - (t + r) \sim (r + s) - (r + t)$ for all $s, t \in S$.

To show that these alternate constructions of $G(S)$ are indeed equivalent, define the map between them given by $[r - s] \mapsto [r] - [s]$. This map is well-defined because if $[r_1 - s_1] = [r_2 - s_2]$ then for some $t \in S$, $r_1 + s_2 + t = r_2 + s_1 + t$. Thus

$$[r_1] - [s_1] + [t] = ([r_1] + [s_2] + [t]) - [s_2] - [s_1] = ([r_2] + [s_1] + [t]) - [s_1] = [r_2] - [s_2] + [t],$$

which gives that $[r_1] - [s_1] = [r_2] - [s_2]$. Finally, since $[r - 0] + [s - 0] = [(r + s) - 0]$, our map sends $[r - 0] \mapsto [r]$, and is clearly an isomorphism.

With these definitions, we can now give the following definition for $K_0(R)$.

Definition 1.1. *Let R be a ring with unit. Then define* $\boldsymbol{K_0(R)}$ *to be the abelian group $G(Proj(R))$, the group completion of $Proj(R)$.*

The construction of K_0 gives us that $K_0(R)$ is *functorial* in R, because a ring homomorphism $\varphi : R \to S$ induces a map of projective modules $P \mapsto S \otimes_\varphi P$, which is considered as a left S-module. Because tensor products commute with direct sums, we have that $(P_1 \oplus P_2) \mapsto (P_1 \oplus P_2) \otimes_\varphi S \cong (P_1 \otimes_\varphi S) \oplus (P_2 \otimes_\varphi S)$. So φ induces a map of semigroups $Proj(R) \to Proj(S)$, which induces a map $Proj(R) \to G(Proj(S))$, and finally, by the universal property of $G(Proj(R)) = K_0(R)$, induces a unique map $\varphi_* : K_0(R) \to K_0(S)$.

With that in mind, if R is a ring with unit, then there exists a unique ring homomorphism $\iota : \mathbb{Z} \to R$ which sends $1 \in \mathbb{Z}$ to the unit in R. This induces a map $\iota_* : K_0(\mathbb{Z}) \to K_0(R)$.

Definition 1.2. *The* ***reduced*** K_0-***group*** *of R is the quotient $\tilde{K_0}(R) := K_0(R)/\iota_*(K_0(\mathbb{Z}))$*

In a sense, the reduced K_0-group of R gives the "nonobvious" parts of $K_0(R)$ [2], in an analogous way that the reduced homology of a topological space gives the "nonobvious" information about the structure of a space by omitting the information corresponding to the homology of a single point. We would expect that in some uninteresting cases, $\tilde{K_0}(R)$ would vanish, in an analogous fashion to contractible topological spaces.

Proposition 1.2. *If R is a PID, then $K_0(R) \cong \mathbb{Z}$. Furthermore, we have that the map $\iota_* : K_0(\mathbb{Z}) \cong \mathbb{Z} \to K_0(R)$ is an isomorphism, and thus $\tilde{K_0}(R)$ vanishes.*

Proof. It is a well-known fact that if R is a PID, then any projective module over R has a well-defined rank, and any two projective modules of the same rank are isomorphic. Thus $Proj(R) \cong \mathbb{N}$, and so $G(Proj(R)) \cong G(\mathbb{N}) \cong \mathbb{Z}$.

Then under the map $\iota : \mathbb{Z} \to R$, $\mathbb{Z}^n \mapsto R \otimes_\iota \mathbb{Z}^n \cong R^n$, and so ι preserves rank in the induced map $Proj(\mathbb{Z}) \to Proj(R)$. Thus the induced map $\iota_* : \mathbb{Z} \to \mathbb{Z}$ maps $n \mapsto n$, and ι_* is an isomorphism, as desired. □

In analogue to topological K-theory, $K_0(R)$ has the additional structure of a ring if R is commutative. In this case, any (left) $R-$module is also a right $R-$module, and so it makes sense to think of the tensor product $P \otimes_R Q$ of two projective $R-$modules P and Q. Additionally, if P and Q are finitely generated projective $R-$modules such that $P \oplus P' \cong Q \oplus Q' \cong R^n$ for some n, then

$$(P \otimes_R Q) \oplus ((P \otimes Q') \oplus (P')^n) =$$

$$P \otimes_R (Q \oplus Q') \oplus P'^n = P \otimes_R R^n \oplus P'^n = P^n \oplus P'^n = (P \oplus P')^n = R^{n^2},$$

and so $P \otimes_R Q$ is also projective. Since $P \otimes_R Q$ is also finitely generated, and $\otimes_R$ is well-defined on isomorphism classes of projective modules, we have that $Proj(R)$ is closed under $\otimes_R$, the multiplication operation. Then because $P \otimes_R Q \cong Q \otimes_R P$, when we construct $K_0(R)$ using the construction from Proposition 1.1 and require that $[P] \otimes (-[Q]) = -[P \otimes Q]$ in $K_0(R)$, we have that $K_0(R)$ is a commutative ring. Furthermore, since $R \in Proj(R)$ satisfying

$$R \otimes_R P \cong P \otimes_R R \cong P$$

for all $P \in Proj(R)$, $K_0(R)$ is a commutative ring with unit $[R]$.

1.1 Alternate Definition of K_0 with Idempotent Matrices

There's a final construction of $K_0(R)$ which will be a useful way to think of $K_0(R)$ in certain situations, and which will be referenced in both the definition of $K_1(R)$ and $K_2(R)$. This construction comes by considering $n \times n$ matrices with coefficients in R.

In particular, any projective $R-$module P (such that $P \oplus Q \cong R^n$ for some n) can be thought of as the idempotent matrix M_P corresponding to the projection $R^n \cong P \oplus Q \twoheadrightarrow P$. And if $P_1 \cong P_2$ are two isomorphic, projective $R - modules$, the corresponding matrices M_{P_1} and M_{P_2} only differ by a "change of basis" (of course, the term "basis" is not always well-defined for an arbitrary ring) or, in other words, conjugation by some invertible $n \times n$ matrix.

Using this correspondence we can actually identify $Proj(R)$ with the set of idempotent matrices with coefficients in R, up to conjugation by invertible matrices. To that end, we have the following definitions and results:

Definition 1.3. *Let $GL(n, R)$ be the group of invertible $n \times n$ matrices with coefficients in R. We can embed $GL(n, R)$ into $GL(n + 1, R)$ by mapping any matrix $M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$, for $M \in GL(n, R)$. Of course, this embedding preserves invertability of M. This gives us a directed system*

$$\cdots \hookrightarrow GL(n-1, R) \hookrightarrow GL(n, R) \hookrightarrow GL(n + 1, R) \hookrightarrow \cdots$$

and we let $GL(R)$ be the direct limit of this directed system. Alternatively, $GL(R)$ is the infinite direct sum $\bigoplus_{n\in\mathbb{N}} GL(n, R)$, where each M is identified with its image under all the embedding maps.

Definition 1.4. *In an analogous manner to Definition 1.3, we can define $M(n, R)$ to be the set of $n \times n$ matrices of R, and we can embed $M(n, R)$ into $M(n+1, R)$ by the map $M \mapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$. As before, this gives us a directed system, and we let $M(R)$ be the direct limit of that system.*

Finally, let Idem(R) be the set of all idempotent matrices in $M(R)$.

Note that any matrix in both $GL(R)$ and $M(R)$ can be considered as a finite matrix. Also, note that $GL(R)$ acts on $Idem(R)$ by conjugation. Now, with these definitions, we can give the final characterization of $K_0(R)$, as follows:

Theorem 1.1. *(Rosenberg, Theorem 1.2.3) For any ring R, $Proj(R)$ may be identified with the set of conjugation orbits of $GL(R)$ on $Idem(R)$. $K_0(R)$ is the Grothendieck group of this semigroup.*

This result is not central to the main goal of this paper, and is included only to give intuition, so we will omit the proof. We invite the curious reader to read the proof of Lemma 1.2.1 and Theorem 1.2.3 of [2].

Example 1.1. *Theorem 1.1 immediately gives that $K_0(R)$ is isomorphic to $K_0(M_n(R))$ for all n, because via the construction of the groups $M(R)$ and $M(M_n(R))$ above, $M(R) \cong M(M_n(R))$.*

2 The Functor K_1

We will now proceed to define $K_1(R)$. To do so, we will make use of Definitions 1.3 and 1.4, which gave us $GL(R)$ and $M(R)$. The definition of both $K_1(R)$ and $K_2(R)$ make use of the same structure that Theorem 1.1 used to give a definition of $K_0(R)$. We will only need one additional definition of a subgroup of $M(R)$ in order to define $K_1(R)$.

Definition 2.1. *We will call a matrix $E \in M(n, R)$* ***elementary*** *if it has $1's$ on the diagonal and at most one nonzero entry off the diagonal. If $E_{ij} = a \in R$ is the nonzero, off-diagonal entry of E, then we will write E as $e_{ij}(a)$ (the identity matrix is simply denoted e). Then let $E(n, R)$ be the subgroup of $M(n, R)$ generated by these matrices. Note that there is a natural embedding $E(n, R) \hookrightarrow E(n+1, R)$ which, maps $e_{ij}(a) \mapsto \begin{pmatrix} e_{ij}(a) & 0 \\ 0 & 1 \end{pmatrix}$.*

As before, this gives us a directed system. Let $E(R)$ be the direct limit of this system, and call $E(R)$ the ***subgroup of elementary matrices****.*

Remark: $E(R)$ contains matrices which are not of the form $e_{ij}(a)$, so calling $E(R)$ the subgroup of elementary matrices is, admittedly, a slight abuse of terminology.

We will end up defining $K_1(R)$ to be the quotient $GL(R)/E(R)$, but to do so we need first to cover some more ground. Namely, we need to prove that $E(R)$ is normal in $GL(R)$. In order to do so, we will prove the result that $[GL(R), GL(R)] = [E(R), E(R)] = E(R)$, which will not only give us that $E(R)$ is normal, but that $E(R)$ is perfect.

We begin with a lemma:

Lemma 2.1. *Let $M \in GL(R)$ be upper-triangular. Then $M \in E(R)$*

Proof. Here we consider M as an $n \times n$ matrix, for some n, as a representative of its equivalence class in $GL(R)$. Note that for all i, j such that $1 \leq i, j \leq n$, the matrix $e_{ij}(M_{ij}) \in E(R)$. Furthermore, we can write

$$M = e_{n-1n}(M_{n-1n}) \cdots e_{2n}(M_{2n}) \cdots e_{23}(M_{23}) e_{1n}(M_{1n}) \cdots e_{13}(M_{13}) e_{12}(M_{12}). \qquad (1)$$

So M is a product of elementary matrices, and thus $M \in E(R)$. This also covers the case in which M is lower-triangular, because we can do the construction above for M^T, and then the transpose of Equation 1 gives us our decomposition of M in elementary matrices. □

Corollary 2.1.1. *Let $A \in GL(n, R)$. Then $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in E(2n, R)$.*

Proof. Note that

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

.

The result follows because each of the matrices on the right-hand side is either upper or lower triangular, and thus in $E(R)$ by Lemma 2.1. □

Proposition 2.1 (Whitehead's Lemma)**.** *Let R be a ring. Then*

$$[GL(R), GL(R)] = [E(R), E(R)] = E(R).$$

Proof. It is a straightforward fact to check that $e_{ik}(ab) = e_{ij}(a)e_{jk}(b)e_{ij}(a)^{-1}e_{jk}(b)^{-1}$ as long as i, j and k are distinct. Thus any elementary matrix $e_{ik}(a)$ is the commutator $[e_{ij}(a), e_{jk}(1)]$, when $j \neq i, k$, and so every generator of $E(R)$ is the commutator of two other generators. So $E(R) \subseteq [E(R), E(R)]$. By definition, $[E(R), E(R)] \subseteq E(R)$, and so $E(R) = [E(R), E(R)]$ (i.e. $E(R)$ is perfect).

Now, note that $[E(R), E(R)] \subseteq [GL(R), GL(R)]$ because $E(R) \subseteq GL(R)$. To show the other inclusion, let $A, B \in GL(n, R)$, and consider the embedding of $ABA^{-1}B^{-1}$ into $GL(2n, R)$ via $ABA^{-1}B^{-1} \mapsto \begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1_n \end{pmatrix}$, and note that:

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1_n \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

Corollary 2.1.1 gives us that each matrix on the right-hand side lies in $E(2n, R)$, and so $ABA^{-1}B^{-1} \in E(R)$, as desired. □

Because $E(R) = [GL(R), GL(R)]$, $E(R)$ is normal in $GL(R)$, so it makes sense to take the quotient $GL(R)/E(R)$. Also, since $[GL(R), GL(R)] = E(R)$, $GL(R)/E(R)$ is the maximal abelian quotient $GL(R)_{ab}$ of $GL(R)$, or the *abelianization* of $GL(R)$. With Proposition 2.1 under our belts, we are prepared to give a definition of $K_1(R)$.

Definition 2.2. *Let R be a ring (with unit). We define $K_1(R) \coloneqq GL(R)/E(R)$.*

As stated before, one way to think of $K_1(R)$ is as the abelianization of $GL(R)$. Another way to think of $K_1(R)$ is as a quotient of $GL(R)$ by the relation that $M_1 \sim M_2$ if M_2 can be produced from M_1 by multiplication from elementary matrices. This is similar to an equivalence via row-reduced echelon form, depending on the ring, because left multiplication by $e_{ij}(a)$ is the familiar row-operation of adding a times the j^{th} row to the i^{th} row.

As before with $K_0(R)$, K_1 is a functor from rings to abelian groups, because any (unit-preserving) map $\varphi : R \to S$ induces a map $GL(R) \to GL(S)$ and $E(R) \to E(S)$, giving us a map $GL(R)/E(R) \to GL(S)/E(S)$.

3 K_0 and K_1 of a Category with Exact Sequences

We have thus far given definitions of both K_0 and K_1 in ring-theoretic terms. Unfortunately, our definitions do not readily generalize to other settings in any obvious way. We would like to think of these groups in a slightly more general setting, in particular because we would like to show the connection between algebraic and topological $K-$theory.

The goal of this section is to lay the groundwork for a suitable generalization of K_0 and K_1 to *categories with exact sequences*. It will be somewhat straightforward to recover from this generalization our original definitions of $K_0(R)$ and $K_1(R)$ of a ring R with unit. However, the new structure will allow us to make the connection between algebraic $K-$theory and topological $K-$theory. By choosing the right category, we will be able to recover $K_0(X)$, the 0^{th} topological $K-$group of a compact Hausdorff topological space X.

Then in Section 4 we will prove a much stronger result, namely that there is an isomorphism of categories which relates algebraic $K-$theory and topological $K-$theory, and which shows that algebraic $K-$theory is a suitable generalization of topological $K-$theory.

We will begin with the definition of a category with exact sequences, assuming that the reader is familiar with both additive and abelian categories.

Definition 3.1. *Let $\mathcal{P}$ be a category. We call $\mathcal{P}$ a* ***category with exact sequences*** *if $\mathcal{P}$ is a full additive subcategory of an abelian category $\mathcal{A}$, such that:*

1. *$\mathcal{P}$ is closed under extensions; i.e. if $P_1, P_2 \in \mathcal{P}$ and there exists a short exact sequence*
$$0 \to P_1 \to P \to P_2 \to 0$$
in $\mathcal{A}$, then P is also in $\mathcal{P}$.

2. *$\mathcal{P}$ has a small skeleton, i.e., $\mathcal{P}$ has a full subcategory $\mathcal{P}_0$ which is small, and such that the inclusion $\mathcal{P}_0 \hookrightarrow \mathcal{P}$ is an equivalence of categories.*

Some examples of such categories include the following:

1. Any small abelian category

2. **Proj**R, the category of finitely generated projective $R-$modules, is a category with exact sequences. It is a full subcategory of the abelian category $R-$**mod**, and is closed under extensions because every short exact sequence splits, and the direct sum of two objects in **Proj**R is an object in **Proj**R. Its small skeleton the set of direct summands of R^n for any $n \in \mathbb{N}$, because any projective $R-$module is isomorphic to a direct summand of R^n for some n, and so the inclusion of this subcategory into **Proj**R is an equivalence of categories. But notice that **Proj**R is, in general, *not* an abelian category since it's not always the case that the cokernel of a map of projective modules is again projective. For example, **Proj** $\mathbb{Z}$ is not abelian, because the cokernel of $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is not an object in **Proj** $\mathbb{Z}$.

3. If X is a compact Hausdorff space, then **Vect**X, the category of vector bundles over X, is also a category with exact sequences because **Vect**X is *isomorphic* to **Proj**$C(X)$, where $C(X)$ is the ring of continuous $k-$valued functions on X, for $k = \mathbb{R}$ or $\mathbb{C}$. Section 4 will be devoted to proving this isomorphism of categories.

The purpose of defining categories with exact sequences is to define some more general notion of K_0 and K_1 of a category. Recall that when we were defining K_0 of a ring R, we looked at *Proj(R)* and found that it was a semigroup under the direct sum operation. We would like to define something analogous to that in the case of the category $\mathbf{Proj}(R)$ which will produce an abelian group from any category with exact sequences.

In the case of $\mathbf{Proj}R$, the statement that $P = P_1 \oplus P_2$ is equivalent to saying that there exists a short exact sequence of the form:

$$0 \to P_1 \to P \to P_2 \to 0$$

or that P is an extension of P_2 by P_1. It is this notion for "addition" that we will use for the more general case of defining K_0 of some category.

Definition 3.2. *Let $\mathcal{P}$ be a category with exact sequences with a small skeleton $\mathcal{P}_0$. We define $\boldsymbol{K_0(\mathcal{P})}$ to be the free abelian group generated by the objects of $\mathcal{P}_0$, modulo the following relations:*

1. $[P] = [P']$ *iff* $P \cong P'$ *in* $\mathcal{P}$, *and*
2. $[P_1] + [P_2] = [P]$ *iff there exists a short exact sequence of the form:*

$$0 \to P_1 \to P \to P_2 \to 0$$

in $\mathcal{P}$.

Because every short exact sequence splits in $Proj(R)$, we have changed nothing from Definition 1.1, where we originally defined $K_0(R)$ for a ring R.

The definition for $K_1(\mathcal{P})$ is also similar to its previous analogue, although not quite as obviously equal as that of $K_0(\mathcal{P})$.

Definition 3.3. *Consider $\mathcal{P}$ as above. Then we define $\boldsymbol{K_1(\mathcal{P})}$ to be the free abelian group generated by elements of the form (P, α), where $P \in \mathcal{P}_0$ and $\alpha \in AutP$, modulo the following relations:*

1. $[(P, \alpha)] + [(P, \beta)] = [(P, \alpha\beta)]$
2. $[(P_1, \alpha_1)] + [(P_2, \alpha_2)] = [(P, \alpha)]$ *iff there exists a commutative diagram in $\mathcal{P}$ of the form:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{\iota} & P & \xrightarrow{\pi} & P_2 & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_2} & & \| \\
0 & \longrightarrow & P_1 & \xrightarrow[\iota]{} & P & \xrightarrow[\pi]{} & P_2 & \longrightarrow & 0
\end{array}$$

with exact rows (note that the rows are identical).

Proposition 3.1. *Let R be a ring with unit. Then:*

1. $K_0(R) \cong K_0(\boldsymbol{Proj}(R))$, *and the isomorphism is natural.*
2. *Similarly, by natural isomorphism,* $K_1(R) \cong K_1(\boldsymbol{Proj}(R))$.

Proof. (1) This isomorphism comes because, in each case, the definitions are identical. So no proof is needed.

(2) We will construct an isomorphism between $K_1(\boldsymbol{Proj}(R))$ as follows: Note that elements $A \in GL(n, R)$ correspond bijectively to to automorphisms $\alpha \in Aut(R^n)$ via left-multiplication. So we define a map $\varphi : K_1(R) \to K_1(\boldsymbol{Proj}(R))$ given by $[A] \mapsto [(R^n, \alpha)]$.

We'll first check that this is well-defined. To do so, consider $A, B \in GL(n, R)$ which correspond to automorphisms $\alpha, \beta \in Aut(R^n)$. Then note that AB corresponds to $\alpha\beta$ (recall that we are considering A as an automorphism of R^n via left multiplication), and so

$$\varphi([A][B]) = \varphi([AB]) = [(R^n, \alpha\beta)] = [(R^n, \alpha)] + [(R^n, \beta)] = \varphi([A]) + \varphi([B]). \tag{2}$$

Also note that if $[A] = [A']$, then A and A' differ only by elements of $E(R)$. So to check that φ is well-defined on equivalence classes, we need only check that if $E \in E(R)$, then $\varphi(E) = 0$ because of Equation 2 above.

To show that $\varphi(E(R)) = 1$, it suffices to show that φ maps the generators of $E(R)$ to 1. This comes immediately by part 2 of Definition 3.3 because we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^{n-1} & \xrightarrow{\iota} & R^n & \xrightarrow{\pi} & R & \longrightarrow & 0 \\ \| & & \| & & \downarrow{\scriptstyle e_{ij}(a)} & & \| & & \| \\ 0 & \longrightarrow & R^{n-1} & \xrightarrow[\iota]{} & R^n & \xrightarrow[\pi]{} & R & \longrightarrow & 0 \end{array}$$

where $e_{ij}(a)$ is a generator of $E(R)$ which can be considered as an element of $E(n, R)$ from the direct limit. (By definition, any matrix in $E(R)$ can be considered as a finite matrix.) Since replacing the $e_{ij}(a)$ arrow in the above diagram with an isomorphism also makes the diagram commute, by part 2 of Definition 3.3, $[(R^n, e_{ij}(a)] = [(R^n, Id_{R^n})]$, and we have that φ is well-defined.

We must now show that φ is an isomorphism, and we will begin with surjectivity. To avoid confusion, we warn the reader that the group operation in $K_1(R)$ will be denoted multiplicitavely, as $K_1(R)$ is the quotient of a matrix group, and that the operation in $K_1(\boldsymbol{Proj}(R))$ will be denoted additively.

So to show surjectivity, consdier $(P, \alpha) \in K_1(\boldsymbol{Proj}(R))$, where $P \in \boldsymbol{Proj}(R)$ such that $P \oplus Q \cong R^n$. Then note that $[(P, \alpha)] + [(Q, id_Q)] = [(P \oplus Q, \alpha \oplus id_Q)] = [(R^n, \alpha \oplus id_Q)]$. Thus $[(P, \alpha)] + [(Q, id_Q)]$ lies in the image of φ. But $[(Q, id_Q)]$ is the identity element in $\boldsymbol{Proj}(R)$, and so $[(P, \alpha)]$ lies in the image of φ, and φ surjects, as desired.

Finally, we show injectivity. To do so, suppose that $\varphi([C]) = 0$ for some $C \in GL(n, R)$. Then if γ is the automorphism of R^n corresponding to C, we have that $[(R^n, \gamma)]$ lies in the subgroup of $K_1(\boldsymbol{Proj}(R))$ generated by the relations from the definition of $K_1(R)$, namely:

$$[(P, \alpha)] + [(P, \beta)] = [(P, \alpha\beta)] \tag{3}$$

$$[(P, \alpha)] = [(P_1, \alpha_1)] + [(P_2, \alpha_2)]. \tag{4}$$

This gives, as it did earlier in this proof, that

$$[(P, \alpha)] = [(P \oplus Q, \alpha \oplus id_Q)]$$

where $P \oplus Q \cong R^n$, and so we can assume that $[(R^n, \gamma)]$ lies in the subgroup of $K_1(\boldsymbol{Proj}(R))$ generated by relations associated to finitely generated free modules.

We take our finitely generated free modules to run over the set $\{R^n : n \in \mathbb{N}\}$, and identify each automorphism of a free module with its matrix. Then in the free abelian group F generated by $[A, j]$, with $A \in GL(j, R)$ and $j \in \mathbb{N}$, $[C, n]$ lies in the subgroup generated by

$$[A, j] + [B, j] = [BA, j], \tag{5}$$

and

$$[A, j + k] = [A_1, j] + [A_2, k] \tag{6}$$

Where (5) corresponds to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^j & \longrightarrow & R^{j+k} & \longrightarrow & R^k & \longrightarrow & 0 \\ \Vert & & {\scriptstyle A_1}\downarrow & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A_2} & & \Vert \\ 0 & \longrightarrow & R^j & \longrightarrow & R^{j+k} & \longrightarrow & R^k & \longrightarrow & 0 \end{array}$$

and the relations in (5) and (6) come, respectively, from the relations in (3) and (4).

We can rewrite the relation in (5) as linear combinations of those of either:

$$[A, j] = [BAB^{-1}, j], \tag{7}$$

which corresponds to the case that $k = 0$ and allows for arbitrary changes of basis, or

$$[A_1 \oplus A_2, j + k] = [A_1, j] + [A_2, k], \tag{8}$$

where $A_1 \oplus A_2$ denotes the matrix $\begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix}$, and the relation from (8) corresponds to the standard inclusion $R^j \hookrightarrow R^{j+k}$. The quotient of the free abelian group F by the subgroup generated by the relations from (5) and (6) gives us the direct sum $\bigoplus_j GL(j, R)_{ab}$. Then dividing by the subgroup generated by the relations from (7) and (8) gives us $\varinjlim GL(j, R)_{ab} = GL(R)_{ab} = K_1(R)$, modulo the additional relation that $\left[\begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix}\right] = \left[\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\right]$ because they both can satisfy the same commutative diagram from part 2 of Definition 3.3. But elements of $K_1(R)$ already satisfy this relation, so $[C] = 1$ in $K_1(R)$, and φ injects, giving us our desired result. $\square$

4 Equivalence of Categories and Topological $K-$Theory

In this section we will refer to Swan's theorem, an important application of algebraic $K-$theory to topological $K-$theory which shows that algebraic $K-$theory, as we have defined it here, is a suitable generalization of topological $K-$theory. This is result, as referenced in Section 3, is an isomorphism of categories between **Proj**R, the category of finitely generated projective $R-$modules, and **Vect**X, the category of vector bundles over X, a compact Hausdorff topological space.

For the sake of completeness, we will give a brief introduction to vector bundles, along with some examples, but assume that the reader is familiar with elementary topological $K-$theory.

Definition 4.1. *Let X be a compact Hausdorff topological space. A* ***vector bundle*** *over X is a space E and a map $E \xrightarrow{p} X$ such that for all $x \in X$:*

1. *$p^{-1}(x) \cong V$ for some finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, and*

2. *there exists some open set $U_x \ni x$ such that $p^{-1}(U_x) \cong U_x \times V$.*

We usually say that E the ***total space****, and X the* ***base space****.*

Example 4.1. *Let $X = S^2$ and let $E = \{(x, y) \in X \times \mathbb{R}^3 : y$ is in the tangent space of S^2 at $x\}$. Let $p : E \to X$ be the projection map onto the first coordinate. Then for all $x \in X$, $p^{-1}(x)$ is homeomorphic to the tangent plane of S^2 at x, so $p^{-1}(x) \cong \mathbb{R}^2$. Also, for any open set $U \subsetneq S^2$ not containing all of S^2, $p^{-1}(U)$ is homeomorphic to $U \times \mathbb{R}^2$. Thus $E \xrightarrow{p} X$ is a vector bundle over X.*

Example 4.2. *Let $X = S^1$. We could construct a vector bundle analogously to that in Example 4.1 by just replacing S^2 with S^1, and $\mathbb{R}^2$ with $\mathbb{R}$, and noticing that everything still holds. However, there is a more interesting bundle of S^1, called the Möbius bundle. The total space of this bundle can be constructed by taking $\mathbb{R} \times [0, 1]$ and identifying $(0, x)$ with $(1, -x)$. Then taking the projection map to be projection onto the first coordinate, we have a map $E \xrightarrow{p} S^1$ such $p^{-1}(x) \cong \mathbb{R}$ for all $x \in S^1$, and any open set $U \subsetneq S^1$ not containing S^1 has the property that $p^{-1}(U) \cong U \times \mathbb{R}$. Thus $E \xrightarrow{p} X$ is a vector bundle over X.*

Definition 4.2. *Let X be a compact Hausdorff topological space, and suppose that $E \xrightarrow{p} X$ is a vector bundle over X. A* ***continuous section*** *of p is a continuous map $s : X \to E$ such that $p \circ s = id_X$.*

Throughout the rest of this paper, we will use the terms section and continuous section interchangeably.

Proposition 4.1. *Let X be a compact Hausdorff topological space, and suppose that both $E_1 \xrightarrow{p_1} X$ and $E_2 \xrightarrow{p_2} X$ are vector bundles over X. Let (E, p) be the pullback of*

$$\begin{array}{ccc} & & E_2 \\ & & \downarrow \\ E_1 & \longrightarrow & X \end{array}$$

Then (E, p) is a vector bundle over X, which we will denote, with slight abuse of notation, as $E_1 \oplus E_2$.

Proof. Let $x \in X$, and first note that $p^{-1}(x) \cong p_1^{-1}(x) \oplus p_2^{-1}(x)$ by properties of the pullback. Since for some finite-dimensional vector spaces V_1 and V_2, $p_1^{-1}(x) \oplus p_2^{-1}(x) \cong V_1 \oplus V_2$, it follows that $p^{-1}(x) \cong V_1 \oplus V_2$.

Then let $U_1, U_2 \ni x$ be open sets in X such that $p_1^{-1}(U_1) \cong U_1 \times V_1$ and $p_2^{-1}(U_2) \cong U_2 \times V_2$, and note that, for $U = U_1 \cap U_2$, it still holds that $p_1^{-1}(U) \cong U \times V_1$ and $p_2^{-1}(U) \cong U \times V_2$ because both $\mathbb{C}^n$ and $\mathbb{R}^n$ are homeomorphic to any connected open subset. Then by properties of the pullback, $p^{-1}(U) \cong U \times (V_1 \oplus V_2)$.

Thus $E \xrightarrow{p} X$ is a vector bundle, as desired. □

[Remark: Of course, $E_1 \oplus E_2$ is not the actual direct sum of topological spaces E_1 and E_2, because the direct sum of topological spaces is the disjoint union. However, on each fiber of X, $E_1 \oplus E_2$ is indeed the direct sum of vector spaces $p_1^{-1}(x)$ and $p_2^{-1}(x)$. Since we do not generally use the direct sum notation in topology, it is fitting to employ it in this more linear-algebraic sense.]

Since vector bundles are closed under addition, the set of isomorphism classes of finite-dimensional vector bundles over X, denoted $\text{Vect}(X)$, forms a commutative semigroup ($E_1 \oplus E_2 \cong E_2 \oplus E_1$ canonically), with unit the zero bundle, $X \times \{*\} \xrightarrow{p} X$. Then as in Section 1, we can take the group completion $G(\text{Vect}(X))$ of $\text{Vect}(X)$, and we have an abelian group. This is precisely the definition of the 0^{th} topological $K-$group, as follows:

Definition 4.3. *Let X be a compact Hausdorff topological space. Then the 0^{th} topological $K-$group of X, denoted $K_k^0(X)$ is $G(Vect_k(X))$.*

We are finally prepared to state the theorem to which this section is dedicated, and show the connection between algebraic and topological $K-$theory.

Theorem 4.1 (Swan)**.** *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let X be a compact Hausdorff topological space. Let $R = C^{\mathbb{F}}(X)$ be the ring of continuous maps $X \to \mathbb{F}$. If $E \xrightarrow{p} X$ is a $\mathbb{F}-$vector bundle over X, let*

$$\Gamma(X, E) = \{s : X \to E : s \textit{ is a section of } p\}$$

be the set of continuous sections of p. Note that $\Gamma(X, E)$ is an $R-$module. Then $\Gamma(X, E)$ is finitely generated and projective over R, and every finitely generated projective module over R arises (up to isomorphism) from this construction. Thus the map $E \mapsto \Gamma(X, E)$ induces an isomorphism of categories from the category of vector bundles over X to the category of finitely generated projective $R-$modules.

Proof. We will first show that $\Gamma(X, E)$ is finitely generated. To that end, let $E \xrightarrow{p} X$ be a vector bundle over X and define $\Gamma(X, E)$ as above. Then for each $x \in X$ there exists some open neighborhood $U_x \ni x$ such that $p^{-1}(U_x) \cong X \times V$ for some finite-dimensional vector space V. If V has dimension n, then the n sections $e_i : U_x \to V$ which maps $U_x \mapsto e_i$, the i^{th} basis vector of V, generate the sections of the trivial bundle $E \xrightarrow{p} U_x$ (the bundle $E \xrightarrow{p} X$ restricted to U_x).

Since X is compact, we can cover X with a finite collection of such open sets, indexed U_i, and choose a partition of unity (f_i) subordinate to the covering (i.e., for all i and all $x \in X$, $0 \leq f_i(x) \leq 1$, $f_i(x) = 0$ if $x \notin U_i$, and for all $x \in X$, $\sum_i f_i(x) = 1$).

Then we make a generating set for $\Gamma(X, E)$ out of sections $e_{ij} := f_i e_j$. Each of these sections is supported in U_i and can extend to all of X by simply defining $e_{ij}(x) = 0$ if $x \notin U_i$. By their construction, the e_{ij}'s generate $\Gamma(X, E)$ as an $R-$module, and thus $\Gamma(X, E)$ is finitely generated.

We will now show that $\Gamma(X, E)$ is projective over R. To do this, choose generators s_j of $\Gamma(X, E)$ (which could be the same ones we constructed above), and consider the trivial bundle $X \times \mathbb{F}^k \xrightarrow{\pi_i} X$. We will construct a morphism to the total space, $\mathbb{F}^k \xrightarrow{\varphi} E$, defined by:

$$(x, v_1, ..., v_k) \mapsto \sum_{j=1}^{k} v_j s_j(x).$$

Recall that the s_j's span $\Gamma(X, E)$, so they span $p^{-1}(x)$ for each x, and thus φ is surjective on each fiber. We define a subbundle E' of given by $E' = \ker\varphi$, where each fiber E'_x of $x \in X$ is given by $\ker\varphi_x$. This is a vector bundle because, $\ker\phi_x$ is a finite-dimensional vector space, and if E is trivial over $U \subseteq X$, then $\varphi|_{\pi^{-1}(U)} : X \times \mathbb{F}^k \to p^{-1}(U)$ is a linear map of vector spaces, and thus $\ker\varphi|_{\pi^{-1}(U)} \cong X \times V$ for some finite-dimensional vector space V. Thus E' restricted to U is also the trivial bundle.

Then note that, if $E \oplus E' \cong X \times \mathbb{F}^k$, then

$$\Gamma(X, E) \oplus \Gamma(X, E') \cong \Gamma(X, X \times \mathbb{F}^k) \cong R^k$$

and thus $\Gamma(X, E)$ is projective (note that $\Gamma(X, X \times \mathbb{F}^k) \cong R^k$ because k continuous, $\mathbb{F}-$valued functions uniquely define a section of X, and each section of X is likewise given by a $k-$tuple of continuous, $\mathbb{F}-$valued functions).

We will do this by introducing an inner product on E. (Recall that an inner product can be introduced on a vector bundle if X is paracompact, because each local trivialization is equipped with an inner product, and these can all be patched together by using a partition of unity [1]. Since X is compact, and thus paracompact, we have a well-defined metric on all vector bundles of X, in particular on E and E'.)

Thus we have a metric on both E and on $X \times \mathbb{F}^k$ which comes from the standard inner product on $\mathbb{F}^k$. Also, with respect to this metric φ has an adjoint, $\varphi^* : E \to X \times \mathbb{F}^k$ such that $\langle \varphi v, w \rangle = \langle v, \varphi^* w \rangle$. Since φ is surjective on each fiber of $x \in X$, φ^* is injective, and $\text{Im}(\varphi^*)$ is the orthogonal complement of E', $E'^{\perp}$. Thus φ^* is an isomorphism $E \cong E'^{\perp}$, and it is a basic result in topological $K-$theory that $E \oplus E^{\perp}$ is the trivial bundle for any vector bundle E over a paracompact Hausdorff topological space [1]. Thus $E \oplus E'$ is trivial, and $\Gamma(X, E)$ is indeed projective.

Our next task is to show that every finitely generated projective module over R corresponds to $\Gamma(X, E)$ of some vector bundle E over X. To that end, suppose that P is a projective $R-$module such that $P \oplus Q \cong R^n \cong C(X, \mathbb{F}^n)$. Then P is a collection of continuous maps $X \to \mathbb{F}^n$, and so we can define a vector bundle E as

$$E = \{(x, v_1, ..., v_n) \in X \times \mathbb{F}^n : s(x) = (v_1, ..., v_n) \text{ for some } s \in P\}.$$

We claim that if we define $E \xrightarrow{p} X$ to be projection onto the first factor, E is a vector bundle over X, where vector addition and scalar multiplication in each fiber is given precisely from that of $\mathbb{F}^n$.

Then our final item to check is local triviality. Let $x \in X$ and choose elements $e^1, ..., e^r \in P$ such that $e^1(x), ..., e^r(x)$ are a basis for $E_x = p^{-1}(x) \subseteq \mathbb{F}^n$. Since these are vector-valued functions, we will write $e^i = (e^i_1, e^i_2, ..., e^i_n)$. And since the functions $e^1(x), ..., e^r(x)$ are linearly independent, we can choose $1 \leq j_1 < ... < j_r \leq n$ such that $e = \det(M_{ik}) \neq 0$ at x, where $M_{ik} = (e^i_{j_k})$. Then because $P \oplus Q \cong R^n$, we may choose a complementary basis of elements $f^{n-r}, .., f^n$, and, as above, construct a nonzero determinant f using the f^r. Then both e and f are continuous (as each f^r and e^r is continuous, and the determinant map is also continuous), and so we have an open neighborhood U_x of x such that $e, f \neq 0$ on U_x. Then if $y \in U_x$, the $\{e^i(y)\}_i$ and $\{f^i(y)\}$ generate, respectively, rank$-r$ and rank$-(n-r)$ free submodules of P and Q. Thus these span both P and Q, and we have that both P and Q are trivial over U.

So we now have that, up to isomorphism, each projective $R-$module has the form $\Gamma(X, E)$ for some vector bundle E over X. Additionally, a map of sections $\Gamma(X, E_1) \to \Gamma(X, E_2)$ restricts to a linear map on each fiber, defined by the images of the spanning sections of the fiber. This is exactly a morphism of vector bundles, and thus the functor which sends $E \mapsto \Gamma(X, E)$, and a morphism $E_1 \xrightarrow{\varphi} E_2$ to $\Gamma(X, E_1) \xrightarrow{\phi_*} \Gamma(X, E_2)$, given by $s \mapsto \varphi \circ s$, is bijective on objects as well as morphisms, and we have an isomorphism of categories.

□

Corollary 4.1.1. *Theorem 4.1 immediately gives us that $K^0(X) \cong K_0(R)$.*

5 The Functor K_2

We will end our exposition of algebraic $K-$theory by defining $K_2(R)$ for any ring (with unit) R. Intuitively speaking, the definition of $K_2(R)$ will show that $K_2(R)$ measures those relations of elementary matrices of the form $m_{ij}(a)$ (drawing from the definition of $E(R)$ in Section 2) which are "non-obvious," in that they do not correspond exactly with the relations of elements of $E(R)$. This idea is somewhat similar to the definition of $K_1(R)$, which showed that $K_1(R)$ measures the failure of general invertible matrices over R to be expressed as a product of its elementary matrices (elements in $E(R)$).

With this in mind, we define the Steinberg group, as follows:

Definition 5.1. *Recall from Section 2 that E(R) is generated by elementary matrices of the form $e_{ij}(a)$, where $i \neq j$ and $a \in R$. Analogously, we define the matrix $m_{ij}(a)$ to be the $n \times n$ matrix with 1's on the diagonal, $a \in R$ on the i, j slot (thus $i \neq j$), and 0's elsewhere. We define the **Steinberg group** of order n, written $St(n, R)$, to be the free group generated by $n \times n$ matrices of the form $m_{ij}(a)$, modulo the following relations (which the elements $e_{ij}(a)$ of E(R) satisfy as well):*

1. $m_{ij}(a)m_{ij}(b) = m_{ij}(a + b)$
2. $m_{ij}(a)m_{kl}(b) = m_{kl}(b)m_{ij}(a), if j \neq k$ *and* $i \neq l$
3. $m_{ij}(a)m_{jk}(b)m_{ij}(a)^{-1}m_{jk}(b)^{-1} = m_{ik}(ab)$, *if* i, j *and* k *are distinct.*
4. $m_{ij}(a)m_{jk}(b)m_{ij}(a)^{-1}m_{ki}(b)^{-1} = m_{kj}(-ba)$, *if* i, j *and* k *are distinct.*

*As in the definitions of $E(R)$, $M(R)$ and $GL(R)$, we can map $St(n, R) \to St(n + 1, R)$ canonically (although not always injectively) via $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, and we get from this a directed system. The **Steinberg group**, written $St(R)$, is the direct limit of this system.*

It is obvious that there is a canonical projection map $St(R) \twoheadrightarrow E(R)$ which maps $n \times n$ matrices $m_{ij}(a) \mapsto e_{ij}(a)$. What is not obvious *a priori* is that the projection is not an isomorphism in general. Indeed, the generators of $E(R)$ satisfy all the relations stated in Definition 5.1. But depending on the ring, they may satisfy more relations which cannot be derived from the four in the definition of $St(R)$ because $E(R) \leq GL(R)$. Then $K_2(R)$ is a sort of measure of whether or not this occurs, in the sense that if $K_2(R)$ is trivial, then the projection is indeed an isomorphism, and if not, then $K_2(R)$ is the subgroup of $St(R)$ generated by the relations which the generators of $E(R)$ satisfy, and which cannot be derived from the four in the definition. Thus, naturally, we have the following definition of $K_2(R)$:

Definition 5.2. *Let R be a ring with unit. We define $\boldsymbol{K_2(R)}$ to be the kernel of the canonical projection map $St(R) \twoheadrightarrow E(R)$.*

Note that a ring homomorphism $\varphi : R \to S$ induces maps $\varphi_* : E(R) \to E(S)$ and $\varphi_* : St(R) \to St(S)$ via $e_{ij}(a) \mapsto e_{ij}(\varphi(a))$ and $m_{ij}(a) \mapsto m_{ij}(\varphi(a))$, respectively. This gives that the diagram

$$\begin{array}{ccc} St(R) & \xrightarrow{\varphi_*} & St(S) \\ \downarrow & & \downarrow \\ E(R) & \xrightarrow[\varphi_*]{} & E(S) \end{array}$$

commutes, because, along the top, $m_{ij}(a) \mapsto m_{ij}(\varphi(a)) \mapsto e_{ij}(\varphi(a))$, and, along the bottom, $m_{ij}(a) \mapsto e_{ij}(a) \mapsto e_{ij}(\varphi(a))$. Thus under the induced map $\varphi_* : St(R) \to St(S)$, $\varphi_*(ker(St(R) \to E(R)) \subseteq (ker(St(S) \to E(S))$ and we have that φ induces a map $\varphi_* : K_2(R) \to K_2(S)$. So just as in the case of $K_0(R)$ and $K_1(R)$, $K_2(R)$ is functorial in R.

However, it's not at all clear that $K_2(R)$ is abelian, which we would like it to be, as both $K_0(R)$ and $K_1(R)$ are abelian, as well as the topological $K-$groups which this theory generalizes. It is simply defined as the kernel of a group homomorphism between to non-abelian groups, and so *a priori* there is no reason why it should be abelian. We will dedicate the remainder of this section to proving that $K_2(R)$ is indeed abelian. To do this, we will need the following notion:

Definition 5.3. *Let G_1, G_2, G be groups. An extension*

$$e \to G_1 \to G \to G_2 \to e.$$

of G_2 by G_1 is ***central*** *if the map $G_1 \hookrightarrow G$ is the inclusion of the center of G.*

Our goal, then, for the rest of this section is to show that the short exact sequence

$$e \to K_2(R) \to St(R) \to E(R) \to e \tag{9}$$

is a central extension of $E(R)$. Given that, it follows immediately that $K_2(R)$ is an abelian group, just like its counterparts $K_1(R)$ and $K_0(R)$.

As our first step to proving that this extension is indeed central, we prove the following technical lemma:

Lemma 5.1. *Let R be a ring and $n \in N$ such that $n \geq 3$. Let $N(n,R)$ be subgroup of $St(n,R)$ generated by all $m_{ij}(a)$, $a \in R$, with $i < j$. Then φ_n restricted to this subgroup is an isomorphism onto the upper-triangular subgroup of $E(n,R)$. Thus $K_2(R) \cap N(n,R)$ is trivial.*

Proof. Consider $N(n,R)$ as defined above. Then let N_1 be the subgroup of $N(n,R)$ generated by $m_{1j}(a)$ such that $1 < j \leq n$. Note that N_1 is abelian by relation 2 from Definition 5.1 (because $1 \neq j$), and that R^{n-1} surjects onto N_1 via the map:

$$(a_2, a_3, ..., a_n) \mapsto m_{12}(a_2)m_{13}(a_3)\cdots m_{1n}(a_n).$$

Then if φ_n is the projection map $St(n,R) \twoheadrightarrow E(n,R)$, φ_n maps N_1 to the upper-triangular matrices in $E(R)$ whose off-diagonal, nonzero entries are all on the first row. Then under the composition $R^{n-1} \twoheadrightarrow N_1 \overset{\varphi_n}{\to} E(n,R)$, $(a_2, a_3, ..., a_n) \in R^{n-1}$ is mapped to the upper-triangular matrix M in $E(n,R)$ with $M_{1j} = a_j$ and 0 elsewhere above the diagonal, and $R^{n-1} \twoheadrightarrow N_1 \overset{\varphi_n}{\to} E(n,R)$ is injective, giving that φ_n restricted to N_1 is injective.

Then let N_2 be the subgroup of $St(n,R)$ generated by elements of the form $m_{ij}(a)$ sith $i < j$ and $i = 1$ or 2. Then N_2/N_1 is generated by the cosets of elements of the form $m_{2j}(a)$ with $2 < j \leq n$. By the same arguments as before, the group N_2' generated by these elements $m_{2j}(a)$, $2 < j$, is also abelian and an injective image of R^{n-2} via the map

$$(a_3, a_4, ..., a_n) \mapsto e_{23}(a_3)\cdots e_{2n}(a_n),$$

where $e_{23}(a_3)\cdots e_{2n}(a_n) = M$ is the upper-triangular matrix in $E(R)$ with the nonzero entries above the diagonal being $M_{2j} = a_j$, $3 \leq j \leq n$. So φ_n is injective on N_2', and thus

φ_n is injective on N_2, as any element in N_2 can be given as a unique product of an element of N_1 and one of N_2', and no nontrivial element of N_1 the the inverse of an element in N_2'. Repeating this process n times, we have that φ_n is injective on $N_n = N(n,R)$, and thus that φ_n maps $N(n,R)$ isomorphically to the group of upper-triangular $n \times n$ matrices with 1's on the diagonal and entries in R.

Thus $K_2(R) \cap N(n,R)$ is trivial for all n, and we have our result. □

We are now ready to prove our main result of this section, namely:

Theorem 5.2. *The extension from Equation 9 is central.*

Proof. Let $\varphi : St(R) \twoheadrightarrow E(R)$ be the canonical projection map from Equation 9, and let $x \in Z(St(R))$. Then $\varphi(x)$ must commute with $\varphi(y)$. Since φ surjects, that means that $\varphi(x)$ must commute with all $y \in E(R)$. But $E(R)$ has a trivial center since an $n \times n$ matrix M can't commute with each $e_{ij}(1)$ unless M is diagonal and all the entries have the same value. Since each matrix in $E(R)$ can be thought of as an infinite matrix such that, after some $m \in \mathbb{N}$, the entries in the diagonal stabilize to 1's, M must have all 1's on the diagonal, and thus M is the identity matrix in $E(R)$, and $\phi(x) = e$, and $x \in ker(\varphi) = K_2(R)$. Thus $Z(St(R)) \subseteq K_2(R)$, as desired.

Now consider an element $m = m_{i_1j_1}(a_1)\cdots m_{i_nj_n}(a_n) \in K_2(R)$, with $e_{i_1j_1}(a_1)\cdots e_{i_nj_n}(a_n) = 1$ in $E(R)$. Choose some N larger than all the $i_1,...,i_n, j_1,...,j_n$, and note that for any $l \leq n$, $k < N$ and $a \in R$, we have that:

$$m_{i_1j_1}(a_1)m_{kN}(a)x_{i_1j_1}(a_1)^{-1} = \begin{cases} m_{kN}, & k \neq j_1 \\ m_{i_1N}(a_1a)m_{kN}(a), & k = j_1 \end{cases}$$

by our relations in Definition 5.1. Thus m normalizes the subgroup A_N generated by the $m_{kN}(a)$, with $k < N$ and $a \in R$. Since $K_2(R)$ has a trivial intersection with A_N by Lemma 5.1, the restriction of φ to A_N is injective. Since we have that for all $y \in A_N$, $\varphi(mym^{-1}y^{-1}) = \varphi(m)\varphi(y)\varphi(m^{-1})\varphi(y^{-1}) = \varphi(y)\varphi(y^{-1}) = 1$, we get that $mym^{-1}y^{-1}$ must be trivial.

Thus m communtes with any $m_{kN}(a)$ such that N is larger than the indices $i_1,...,i_n, j_1,...,j_n$. Also, since $m_{ij}(a)m_{jk}(b)m_{ij}(a)^{-1}m_{jk}(b)^{-1} = m_{ik}(ab)$ if i,j and k are distinct (relation 3 from Definition 5.1), these matrices generate $St(R)$. Therefore $m \in Z(St(R))$, as desired, and $K_1(R) = Z(St(R))$, which gives our result

□

Corollary 5.2.1. $K_2(R)$ *is an abelian group.*

6 Conclusion

Our goal with this paper was to thoroughly define the three $K-$groups K_0, K_1 and K_2 of a ring, and then to show it in sufficient generality such that the connection between topological and algebraic $K-$theory would be clear. In doing this, we had to omit the long exact sequence of a pair (R, I) of a ring and an ideal, higher $K-$groups, and negative $K-$groups among other things. Given more time and space, these would have been my next priority, after calculating a few good examples (in general, finding $K-$groups require very nontrivial calculations).

References

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